The Cauchy Problem for 1-D Linear Nonconservative Hyperbolic Systems with possibly Expansive Discontinuity of the coefficient: a Viscous Approach.

Bruno Fornet

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B. Fornet[fornet@cmi.univ-mrs.fr] [*†]

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[*]LATP, Université de Provence,39 rue Joliot-Curie, 13453 Marseille cedex 13, France.

[†]LMRS, Université de Rouen,Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray, France.

Abstract

In this paper, we consider nonconservative Cauchy systems with discontinuous coefficients for a noncharacteristic boundary. The considered problems need not be the linearized of a shockwave on a shock front. We introduce then a viscous perturbation of the problem; the viscous solution u^ε depends of the small positive parameter ε. This problem, obtained by small viscous perturbation, is parabolic for fixed positive ε. Under some assumptions, incorporating a sharp spectral stability assumption, we prove the convergence, when $\varepsilon \to 0^+$, of u^ε towards the solution of a well-posed hyperbolic limit problem. Even though the obtained limit problem is well-posed, it is not bound to satisfy a uniform Lopatinski Condition. Our result is obtained, in the 1-D framework, for piecewise constant coefficients. Explicit examples of 2×2 systems satisfying our assumptions are given. They rely on a detailed analysis of our stability assumption (uniform Evans condition) for 2×2 systems.

The obtained result is new and generalizes the scalar expansive case solved in [7], where the considered hyperbolic operator was $\partial_t + a(x)\partial_x$, with $a(x) = a^+ > 0$ if $x > 0$ and $a(x) = a^- < 0$ if $x < 0$. A complete asymptotic description of the layer is given, at any order of approximation. In general, strong amplitude noncharacteristic boundary layers form, which are localized on the area of discontinuity of the coefficient. Characteristic boundary layers, which appear along characteristic curves, also forms. Both type of boundary layers are polarized on specific disjoint linear subspaces.

1 Introduction.

Let us consider the 1-D linear hyperbolic system:

$$\begin{cases} \partial_t u + A(x)\partial_x u = f, & (t,x) \in \Omega, \\ u^\varepsilon|_{t=0} = h & . \end{cases} \tag{1.1}$$

where $\Omega = \{(t,x) \in (0,T) \times \mathbb{R}\}$, with $T > 0$ fixed once and for all. The unknown $u(t,x)$ belongs to $\mathbb{R}^N$ and A belongs to the set of $N \times N$ matrices with real coefficients $\mathcal{M}_N(\mathbb{R})$. A is assumed to satisfy:

$$A(x) = A^+ \mathbf{1}_{x>0} + A^- \mathbf{1}_{x<0},$$

where A^+, A^-, are constant matrices in $\mathcal{M}_N(\mathbb{R})$. As we will detail later, since A is discontinuous through $\{x = 0\}$, this problem has no obvious sense. This problematic relates to many linear scalar works on analogous conservative problems. We can for instance refer to the works of Bouchut, James and

Mancini in [1], [2]; by Poupaud and Rascle in [17] or by Diperna and Lions in [5]. We can also refer to [6] and [7] by Fornet. The common idea is that another notion of solution has to be introduced to deal with linear hyperbolic Cauchy problems with discontinuous coefficients. Note that almost all the papers cited before use a different approach to deal with the problem. Like in [6] and [7], we will opt for a small viscosity approach. Let us describe now the first result obtained in this paper. We consider the following viscous hyperbolic-parabolic problem:

$$\begin{cases} \partial_t u^\varepsilon + A(x)\partial_x u^\varepsilon - \varepsilon \partial_x^2 u^\varepsilon = f, & (t,x) \in \Omega, \\ u^\varepsilon|_{t=0} = h \quad , \end{cases} \tag{1.2}$$

where ε, commonly called viscosity, stands for a small positive parameter. Note well that, if we suppress the terms in $-\varepsilon\partial_x^2$ from our differential operator, the hyperbolic problem obtained has no obvious sense, because of the nonconservative product $A(x)\partial_x u$ not being well-defined when both u and A are discontinuous through $\{x = 0\}$.

The definition of such nonconservative product is of course crucial for defining a notion of weak solutions for such problems. It is an interesting question by itself, solved for a quasi-linear analogous problem by Lefloch and Tzavaras ([13]). Adopting a viscous approach allows us to avoid the difficult question of the definition of the nonconservative product in the linear framework.

In problem (1.2), the unknown is $u^\varepsilon(t,x) \in \mathbb{R}^N$, the source term f belongs to $H^\infty((0,T) \times \mathbb{R})$ and the Cauchy data h belongs to $H^\infty(\mathbb{R})$. We make the classical hyperbolicity assumption, plus we assume the boundary $\{x = 0\}$ is noncharacteristic. In addition, we make a spectral stability assumption, which is an Uniform Evans Condition for a related problem. Last, we make an assumption ensuring that the limit hyperbolic problem satisfied by $u := \lim_{\varepsilon\to 0^+} u^\varepsilon$ is well-posed. A crucial remark is that this limit problem can be reformulated equivalently into a mixed hyperbolic problem on the half-space $\{x > 0\}$, which **does not satisfy a Uniform Lopatinski Condition**. The goal of Proposition 2.10 is to give, for $N = 2$, examples of discontinuities of the coefficient (A^+, A^-) satisfying all our Assumptions. This Proposition relies on explicit algebraic computations of the Evans function performed in the case $N = 2$.

Our assumptions do not forbid A^+ to have only positive eigenvalues and

A^- of to have only negative eigenvalues. In this case, the discontinuity of the coefficient has a completely expansive setting. The question of the selection of a unique solution through a viscous approach was open, for this case, even for $N = 1$, until [7]. Among other things, the result obtained previously in the scalar framework ([7]) is generalized to $N \in \mathbb{N}$ in this paper.

In order to describe our main result, let us introduce some notations. First, Σ is the linear subspace:

$$\Sigma := \left((A^+)^{-1} - (A^-)^{-1}\right) \left(\mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-)\right),$$

where, for instance,

$$\mathbb{E}_-(A^+) = \bigoplus_{\lambda_j^+ < 0} \ker \left(A^+ - \lambda_j^+ Id\right),$$

with λ_j^+ denoting the eigenvalues of A^+, which are real and semi-simple due to the hyperbolicity of the corresponding operator. $\mathbb{I}$ denotes the linear subspace given by:

$$\mathbb{I} := \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+).$$

We choose, once for all, a linear subspace $\mathbb{V}$ such that:

$$\mathbb{E}_-(A^-) + \mathbb{E}_+(A^+) = \mathbb{I} \bigoplus \mathbb{V}.$$

We assume the following:

$$\mathbb{R}^N = \mathbb{I} \bigoplus \mathbb{V} \bigoplus \Sigma.$$

$\Pi_{\mathbb{I}}$ stands then for the linear projector on $\mathbb{I}$ parallel to $\mathbb{V} \bigoplus \Sigma$.

Note that, in [7], as a consequence of our assumptions, we had

$$\mathbb{R}^N = \mathbb{E}_-(A^-) \bigoplus \mathbb{E}_+(A^+) \bigoplus \Sigma,$$

which is the expression of our above assumption in the case $\mathbb{I} = \{0\}$ and also the expression of the uniform Lopatinski Condition in this special case.

This paper is mainly devoted to the proof of the following result: when $\varepsilon \to 0^+$, u^ε converges towards u in $L^2((0,T) \times \mathbb{R})$, where $u := u^+ \mathbf{1}_{x \geq 0} +$

$u^- \mathbf{1}_{x<0}$ is the solution of the following **well-posed, even though not classical**, transmission problem:

$$
(1.3) \qquad \begin{cases} \partial_t u^- + A^- \partial_x u^- = f^-, & (t,x) \in (0,T) \times \mathbb{R}_-^*, \\ \partial_t u^+ + A^+ \partial_x u^+ = f^+, & (t,x) \in (0,T) \times \mathbb{R}_+^*, \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, \\ \partial_x \Pi_{\mathbb{I}} u^+|_{x=0} - \partial_x \Pi_{\mathbb{I}} u^-|_{x=0} = 0, \\ u^-|_{t=0} = h^-, \\ u^+|_{t=0} = h^+. \end{cases}
$$

$f^{\pm}$ and $h^{\pm}$ denotes respectively the restrictions of f and h to $\{\pm x > 0\}$

The proof of our convergence result splits into two parts. First, we construct an approximate solution of our viscous problem (1.2), then, we prove L^2 stability estimates via Kreiss-type Symmetrizers.

2 Nonconservative hyperbolic Cauchy problem with piecewise constant coefficients.

Let us recall the viscous parabolic problem (1.2):

$$
\begin{cases} \partial_t u^\varepsilon + A(x) \partial_x u^\varepsilon - \varepsilon \partial_x^2 u^\varepsilon = f, & (t,x) \in \Omega, \\ u^\varepsilon|_{t=0} = h \quad . \end{cases}
$$

We assume that $A(x) = A^+ \mathbf{1}_{x>0} + A^- \mathbf{1}_{x<0}$, with

Assumption 2.1. *[Hyperbolicity and Noncharacteristic boundary] A^+ and A^- are real diagonalizable constant matrices in $\mathcal{M}_N(\mathbb{R})$, $det A^- \neq 0$ and $det A^+ \neq 0$.*

Since the solution of the parabolic problem (1.2) is continuous, $\partial_x u^\varepsilon$ will not behave as a Dirac measure on $\{x = 0\}$. Moreover, since:

$$
\varepsilon \partial_x^2 u^\varepsilon = f - \partial_t u^\varepsilon - A(x) \partial_x u^\varepsilon,
$$

$\partial_x^2 u^\varepsilon$ got no Dirac measure on $\{x = 0\}$, thus implying the continuity of $\partial_x u^\varepsilon$ through $\{x = 0\}$. As a consequence, we get that u^ε is solution of (1.2) iff

$(u^\varepsilon_R, u^\varepsilon_L)$ is solution of the following transmission problem:

$$
\begin{cases}
\partial_t u^\varepsilon_R + A^+ \partial_x u^\varepsilon_R - \varepsilon \partial_x^2 u^\varepsilon_R = f_R, & \{x > 0\}, t \in (0,T), \\
\partial_t u^\varepsilon_L + A^- \partial_x u^\varepsilon_L - \varepsilon \partial_x^2 u^\varepsilon_L = f_L, & \{x < 0\}, t \in (0,T), \\
u^\varepsilon_R|_{x=0} - u^\varepsilon_L|_{x=0} = 0, & t \in (0,T), \\
\partial_x u^\varepsilon_R|_{x=0} - \partial_x u^\varepsilon_L|_{x=0} = 0, & t \in (0,T), \\
u^\varepsilon_R|_{t=0} = h_R(x), & \{x > 0\}, \\
u^\varepsilon_L|_{t=0} = h_L(x), & \{x < 0\} \quad .
\end{cases}
\tag{2.1}
$$

The subscripts "L" [resp "R"] are used for the restrictions of the concerned functions to the **L**eft-hand side [resp **R**ight-hand side] of the boundary $\{x = 0\}$. We could refer to $\{x = 0\}$ as a boundary since the transmission problem (2.1) can be recast as the doubled problem on a half-space (2.2):

$$
\begin{cases}
\partial_t \tilde{u}^\varepsilon + \tilde{A} \partial_x \tilde{u}^\varepsilon - \varepsilon \partial_x^2 \tilde{u}^\varepsilon = \tilde{f} & \{x > 0\}, t \in (0,T) \\
\tilde{\mathcal{M}} \tilde{u}^\varepsilon|_{x=0} = 0 \\
\tilde{u}^\varepsilon|_{t=0} = \tilde{h}
\end{cases}
\tag{2.2}
$$

where

$$\tilde{u}^\varepsilon(t,x) = \begin{pmatrix} u^\varepsilon_R(t,x) \\ u^\varepsilon_L(t,-x) \end{pmatrix}$$

The new source term writes $\tilde{f} = \begin{pmatrix} f_R(t,x) \\ f_L(t,-x) \end{pmatrix}$, and the new Cauchy data is $\tilde{h} = \begin{pmatrix} h_R(t,x) \\ h_L(t,-x) \end{pmatrix}$, the new coefficient belongs to $\mathcal{M}_{2N}(\mathbb{R})$ and writes:

$$\tilde{A} = \begin{pmatrix} A^+ & 0 \\ 0 & -A^- \end{pmatrix},$$

and the boundary operator writes

$$\tilde{\mathcal{M}} = \begin{pmatrix} Id & -Id \\ \partial_x & \partial_x \end{pmatrix}.$$

Note that the classical parabolicity and hyperbolicity-parabolicity assumptions, see [14] are trivially satisfied here.
Let $\mathbb{A}^\pm$ denote the matrices defined by:

$$\mathbb{A}^\pm = \begin{pmatrix} 0 & Id \\ (i\tau + \gamma) Id & A^\pm \end{pmatrix}.$$

We recall that we denote by $\mathbb{E}_+(\mathbb{A}^\pm)$ [resp $\mathbb{E}_-(\mathbb{A}^\pm)$] the linear subspace spanned by the generalized eigenvectors of $\mathbb{A}^\pm$ associated to the eigenvalues of $\mathbb{A}^\pm$ with positive [resp negative] real part and

$$det\left(\mathbb{E}_-(\mathbb{A}^+(\zeta)), \mathbb{E}_+(\mathbb{A}^-(\zeta))\right)$$

is the determinant obtained by taking orthonormal bases for both $\mathbb{E}_-(\mathbb{A}^+(\zeta))$ and $\mathbb{E}_+(\mathbb{A}^-(\zeta))$. We introduce the weight $\Lambda(\zeta)$ used to deal with high frequencies:

$$\Lambda(\zeta) = \left(1 + \tau^2 + \gamma^2\right)^{\frac{1}{2}}.$$

Let J_Λ be the mapping from $\mathbb{C}^N \times \mathbb{C}^N$ to $\mathbb{C}^N \times \mathbb{C}^N$ $(u, v) \mapsto (u, \Lambda^{-1}v)$. We can introduce now the scaled negative and positive spaces of matrices $\mathbb{A}^\pm$:

$$\tilde{\mathbb{E}}_\pm(\mathbb{A}^\pm) := J_\Lambda \mathbb{E}_\pm(\mathbb{A}^\pm).$$

Our stability assumption writes:

Assumption 2.2. *[Uniform Evans Condition]*
$(\tilde{\mathcal{H}}^\varepsilon, \tilde{\mathcal{M}})$ *satisfies the Uniform Evans Condition which means that, for all* $\zeta = (\tau, \gamma) \in \mathbb{R} \times \mathbb{R}^+ - \{0_{\mathbb{R}^2}\}$, *there holds:*

$$\left| det\left(\tilde{\mathbb{E}}_-(\mathbb{A}^+(\zeta)), \tilde{\mathbb{E}}_+(\mathbb{A}^-(\zeta))\right) \right| \geq C > 0.$$

In a different framework than ours, the study of such stability assumption has been done in many papers. For example, we can refer the reader to the paper of Gardner and Zumbrun ([8]), Guès, Métivier, Williams and Zumbrun ([9]), Métivier and Zumbrun ([15]), Rousset ([18]) and finaly Serre ([19]) . A more recent reference is [3] by Benzoni, Serre and Zumbrun.

Assumption 2.3. *There holds:*

$$\left(\mathbb{E}_-(A^-) + \mathbb{E}_+(A^+)\right) \bigoplus \Sigma = \mathbb{R}^N.$$

Keeping in mind that the linear subspace $\mathbb{I}$ is defined by $\mathbb{I} := \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+)$, Assumption 2.3 also writes:

$$\mathbb{R}^N = \mathbb{I} \bigoplus \mathbb{V} \bigoplus \Sigma. \tag{2.3}$$

We introduce then the projectors associated to this decomposition, that we respectively note: $\Pi_{\mathbb{I}}$, $\Pi_{\mathbb{V}}$ and Π_Σ.

After introducing the necessary notations, we will formulate an assumption concerning the structure of the discontinuity (A^-, A^+).

By assumption 2.1, there are two nonsingular matrices P^+, P^- and two diagonal matrices D^+ and D^- such that $D^+ = (P^+)^{-1}A^+P^+$ and $D^- = (P^-)^{-1}A^-P^-$. We denote then $\mathbb{J} := \mathbb{E}_-(D^-)\bigcap\mathbb{E}_+(D^+)$. Let us choose two linear subspaces of $\mathbb{R}^N$, $\mathbb{V}_1$ and $\mathbb{V}_2$ such that:

$$\mathbb{V}_1 \bigoplus \mathbb{J} = \mathbb{E}_+(D^+),$$

and

$$\mathbb{V}_2 \bigoplus \mathbb{J} = \mathbb{E}_-(D^-).$$

Assumption 2.4 (Structure of discontinuity)**.**
There holds:

$$P^+\mathbb{V}_1 \bigoplus \left(P^+\mathbb{J} + P^-\mathbb{J}\right) \bigoplus P^-\mathbb{V}_2 \bigoplus \Sigma = \mathbb{R}^N$$

Moreover, the mapping

$$M := \begin{pmatrix} \Pi_{\mathbb{I}}P^+(D^+)^{-1} & -\Pi_{\mathbb{I}}P^-(D^-)^{-1} \\ P^+ & -P^- \end{pmatrix}$$

from $\mathbb{J}\times\mathbb{J}$ *into* $\mathbb{I}\times(P^+\mathbb{J}+P^-\mathbb{J})$ *defines an isomorphism between* $\mathbb{J}\times\mathbb{J}$ *and* $\mathbb{I}\times(P^+\mathbb{J}+P^-\mathbb{J})$ *. Finally, we assume that:* $\dim\mathbb{E}_-(A^+)\bigcap\mathbb{E}_+(A^-) = \dim\Sigma$.

Remark 2.5. *If* $\dim\mathbb{I} = \dim\mathbb{J}$, *then Assumption 2.4 implies that* $P^+\mathbb{J} = P^-\mathbb{J}$.

Let us make a remark concerning 2×2 strictly hyperbolic systems. We take $A^- = \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix}$ and $A^+ = \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix}$, with $d_1^- < 0$ and $d_1^+ > 0$ and $\alpha \in \mathbb{R}^*$. We have $P^- = Id$, $P^+ = \begin{pmatrix} 1 & 1 \\ 0 & \frac{d_1^+ - d_2^+}{-\alpha} \end{pmatrix}$, $D^- = A^-$ and $D^+ = \begin{pmatrix} d_1^+ & 0 \\ 0 & d_2^+ \end{pmatrix}$. As a consequence, $\mathbb{J} = Span\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Moreover $\mathbb{V}_2 = \{0\}$ because $\mathbb{J} = \mathbb{E}_-(A^-)$. Since $\mathbb{E}_+(D^+) = \mathbb{R}^2$, we take $\mathbb{V}_1 = Span\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Moreover, $\mathbb{E}_-(A^+)\bigcap\mathbb{E}_+(A^-) = \{0\}$ thus $\Sigma = \{0\}$.
We check then easily that, like before, if we take $d_2^- > 0$ and $d_2^+ < 0$, Assumption 2.4 is not satisfied for any $\alpha \neq 0$. More general examples of this

form will be analyzed thanks to a new assumption about the structure of the discontinuity, that will be introduced now.

The general assumption is Assumption 2.4. However, we also state a special set of sufficient conditions, which are easier to check in some cases. They write:

Assumption 2.6 (Structure of discontinuity, sufficient version)**.**
We assume that:

- $\dim \Sigma = \dim \mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-)$.
- $A^- \mathbb{I} = \mathbb{I}$
- $A^+ \mathbb{I} = \mathbb{I}$
- $\ker((A^+)^{-1} - (A^-)^{-1}) \bigcap \mathbb{I} = \{0\}$
- $\mathbb{E}_-((Id - \Pi_{\mathbb{I}})A^-(Id - \Pi_{\mathbb{I}})) \bigoplus \mathbb{E}_+((Id - \Pi_{\mathbb{I}})A^+(Id - \Pi_{\mathbb{I}})) \bigoplus \Sigma = \mathbb{V} \bigoplus \Sigma$
- $\dim \mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-) = \dim \Sigma$.

Assumption 2.6 is a sufficient condition for Assumption 2.4 to hold. While this assumption is less general than Assumption 2.4, it is in general easier to check.

If A^- has only negative eigenvalues and A^+ has only positive eigenvalues (totally expansive case), this assumption reduces to:

$$\ker\left((A^+)^{-1} - (A^-)^{-1}\right) \bigcap \mathbb{I} = \{0\}.$$

Since $\mathbb{I} = \mathbb{R}^N$ in the totally expansive case, the assumption also writes:

$$det\left((A^+)^{-1} - (A^-)^{-1}\right) \neq 0.$$

Moreover, if both A^+ and A^- are diagonal or if we make the same assumptions as in [7], this assumption trivially holds.
Let us now give an example for which Assumption 2.4 holds for strictly hyperbolic 2×2 systems. Let us take $A^- = \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix}$ and $A^+ = \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix}$, with $d_1^- < 0$, $d_2^- > 0$, $d_1^+ > 0$, $d_2^+ > 0$ and $\alpha \in \mathbb{R}^*$. We assume moreover that the eigenvalues of A^- and A^+ are all distinct.

Note well that there is no lack of generality in considering A^- diagonal since, by change of basis, we can diagonalize either A^- or A^+. We have then $\mathbb{E}_-(A^-) = Span\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{E}_+(A^+) = \mathbb{R}^2$, which implies that: $\mathbb{I} = Span\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We have moreover $A^+\mathbb{I} = A^-\mathbb{I} = \mathbb{I}$. Since $\mathbb{E}_-(A^+) = \{0\}$ and $\mathbb{E}_+(A^-) = Span\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get that $\Sigma = \{0\}$.

Moreover, $\left((A^+)^{-1} - (A^-)^{-1}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{d_1^+} - \frac{1}{d_1^-} \\ 0 \end{pmatrix}$, which implies that:

$$Ker((A^+)^{-1} - (A^-)^{-1})\bigcap \mathbb{I} = \{0\}.$$

Let us take $\mathbb{V} = \mathbb{I}^\perp = Span\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. There holds: $\mathbb{I}\bigoplus\mathbb{V} = \mathbb{R}^2$. We can make this choice whenever $\Sigma = \{0\}$. We have now to check that:

$$\mathbb{E}_-(\Pi_\mathbb{V} A^- \Pi_\mathbb{V})\bigoplus\mathbb{E}_+(\Pi_\mathbb{V} A^+ \Pi_\mathbb{V}) = \mathbb{V}.$$

Let us take $v \in \mathbb{V}$, we have then $v = \Pi_\mathbb{V} v$. $\Pi_\mathbb{V}$ writes:

$$\Pi_\mathbb{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Actually $\Pi_\mathbb{V} A^- \Pi_\mathbb{V} = \begin{pmatrix} 0 & 0 \\ 0 & d_2^- \end{pmatrix}$ and $\Pi_\mathbb{V} A^+ \Pi_\mathbb{V} = \begin{pmatrix} 0 & 0 \\ 0 & d_2^+ \end{pmatrix}$ thus $\mathbb{E}_-(\Pi_\mathbb{V} A^- \Pi_\mathbb{V}) = \{0\}$ and $\mathbb{E}_+(\Pi_\mathbb{V} A^+ \Pi_\mathbb{V}) = \mathbb{V}$, hence we have checked that Assumption 2.4 holds for the considered matrices A^- and A^+. Let us discuss this example further. Firstly, this example works more generally for $sign(d_2^-) = sign(d_2^+)$. Secondly, if we took $d_2^- > 0$ and $d_2^+ < 0$ Assumption 2.4 is not satisfied for any $\alpha \neq 0$, but is satisfied for $\alpha = 0$ independently of the signs of $d_1^\pm$ and $d_2^\pm$. Finally, Assumption 2.4 is satisfied in the completely outgoing case i.e if we take $d_2^- < 0$ and $d_2^+ > 0$.

Remark 2.7. *The uniform Evans condition is a criterion of stability that seems difficult to check. This stability assumption has been studied in several papers as it is central, among other things, in the study of the stability of shockwaves. As mentioned in [8], a sufficient condition for the Evans condition to hold begins difficult to establish for systems with $N \geq 3$. However, for large systems, computational methods have been proposed for this purpose, see [11] for a recent approach.*

We will now state some of our results concerning the study of the Evans Condition. For $N = 2$, we will give very simple sufficient conditions for Evans-stability and Evans-instability.
Without lack of generality, we can assume that A^- is diagonal. We denote then by $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ the normalized eigenvectors of A^+. Let us define $q := \dim \Sigma$.

Proposition 2.8. *For $N = 2$, i.e for 2×2 systems, and whether $q = 0$, $q = 1$, or $q = 2$, the problem associated to the choice of matrices (A^+, A^-) satisfying: $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$ is Evans-Stable (but not necessarily uniformly Evans-stable).*

In the following Proposition, $\lambda_1^\pm$ and $\lambda_2^\pm$ denote the two eigenvalues of $A^\pm$.

Proposition 2.9. *Provided that the matrices (A^+, A^-) are such that: $a, b, c, d > 0$, $bc > ad$ and $\lambda_1^+ = -\lambda_2^+ < 0$, $\lambda_1^- = -\lambda_2^- < 0$; the associated problem is strongly Evans-unstable, in the sense that the Evans function vanishes for some (τ, γ) with $\tau \in \mathbb{R}$ and $\gamma > 0$.*

As a consequence of the stability analysis performed in section 3, there holds:

Proposition 2.10. *Let P denote a nonsingular matrix in $\mathcal{M}_2(\mathbb{R})$, then the matrices A^- and A^+ defined by:*

$$A^- = P^{-1} \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix} P$$

and

$$A^+ = P^{-1} \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix} P$$

with $d_1^- < 0$, $d_1^+ > 0$ and $\alpha \in \mathbb{R} - \{0\}$ satisfy all our assumptions iff either d_2^+ and d_2^- have the same sign or $d_2^- < 0$ and $d_2^+ > 0$.

2.1 Construction of an approximate solution as a BKW expansion.

We will construct an approximate solution of problem (2.1) at any order. This construction will show that, if $\mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+) \neq \{0\}$, weak amplitude characteristic boundary layers forms similarly to [6]. Moreover, if $\mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-) \neq \{0\}$, large noncharacteristic boundary layers forms on the area of discontinuity of the coefficients: $\{x = 0\}$.

Let us note $\Omega_L = \{(t,x) \in (0,T) \times \mathbb{R}^{*-}\}$ and $\Omega_R = \{(t,x) \in (0,T) \times \mathbb{R}^{*+}\}$. $u^\varepsilon_{app,L}$ [resp $u^\varepsilon_{app,R}$] denotes the restriction of the solution to Ω_L [resp Ω_R]. We will construct $u^\varepsilon_{app,L} \in C^1(\Omega_L) \bigcap L^2(\Omega_L)$ and $u^\varepsilon_{app,R} \in C^1(\Omega_R) \bigcap L^2(\Omega_R)$. To that aim, let us first introduce some notations. The matrix A^- [resp A^+] has N_- [resp N_+] negative [resp positive] eigenvalues. Let $\mu_1^-, \dots, \mu_{N_-}^-$ be the negative eigenvalues of A^- sorted by increasing order and $\mu_1^+, \dots, \mu_{N_+}^+$ be the positive eigenvalues of A^+ sorted by decreasing order. We introduce the following partition of Ω_L :

$$\Omega_L = \mathcal{C}_L \bigsqcup \left(\bigsqcup_{j=0}^{N_-} \Omega_L^j \right),$$

where

$$\mathcal{C}_L := \bigcup_{j=1}^{N_-} \left\{ (t,x) \in \Omega_L : x - \mu_j^- t = 0 \right\},$$

$$\Omega_L^0 := \left\{ (t,x) \in \Omega_L : x - \mu_1^- t < 0 \right\},$$

and for all $1 \leq j \leq N_- - 1$

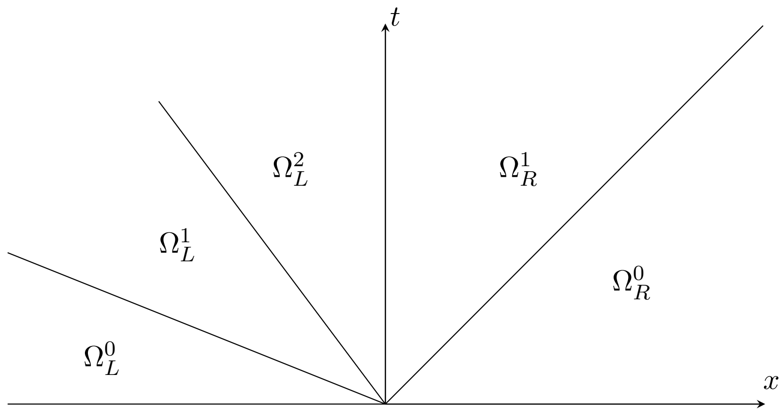
$$\Omega_L^j := \left\{ (t,x) \in \Omega_L : \mu_j^- t < x < \mu_{j+1}^- t < 0 \right\},$$

and

$$\Omega_L^{N_-} := \left\{ (t,x) \in \Omega_L : x - \mu_{N_-}^- t > 0 \right\}.$$

On the right hand side, we do the analogous partition:

$$\Omega_R = \mathcal{C}_R \bigsqcup \left(\bigsqcup_{j=0}^{N_-} \Omega_R^j \right).$$



THIS DRAWING SHOWS THE CASE WHERE $N_- = 2$ AND $N_+ = 1$.

Remark 2.11. *Note that the boundary layer profiles serves the purpose of correcting singularities possibly forming in the small viscosity limit on* $\{x = 0\}$, $\mathcal{C}_R$, *and* $\mathcal{C}_L$. *We will give an ansatz incorporating such terms. In general, it may happen that each line composing* $\mathcal{C}_R$ *and* $\mathcal{C}_L$ *supports singularities of* $u := \lim_{\varepsilon\to 0^+} u^\varepsilon$. *On the other hand, if we take for example* $u|_{t<0} = 0$ *as our Cauchy condition and assume* $f|_{t<0} = 0$, *(which ensures the corner compatibility of our limiting problem), if* $e_j \in \mathbb{V}_2$, *(*e_j *is the* j^{th} *vector of the canonical basis of* $\mathbb{R}^N$*), then* u *has no singularity on* $\{(t,x) \in \Omega_L : x - \lambda_j^- t = 0\}$, *where* λ_j^- *stands for the* j^{th} *diagonal coefficient of* D^-. *The same way, if* $e_j \in \mathbb{V}_1$, *then* u *has no singularity on* $\{(t,x) \in \Omega_R : x - \lambda_j^+ t = 0\}$, *where* λ_j^+ *stands for the* j^{th} *diagonal coefficient of* D^+.

Let us introduce the different profiles and their ansatz. We will construct separately the restriction $u^{\varepsilon,j}_{app,L}$ of $u^\varepsilon_{app,L}$ to each Ω^j_L for $0 \leq j \leq N_-$ so that, the different pieces of approximate solution glued back together gives the approximate solution $u^\varepsilon_{app,L} \in C^1(\Omega_L) \bigcap L^2(\Omega_L)$.

$$u^{\varepsilon,j}_{app,L}(t,x) = \sum_{n=0}^{M} \left(\underline{\mathbf{U}}^j_{n,L}(t,x) + \mathbf{U}^{*,j}_{n,L}\left(t, \frac{x}{\varepsilon}\right)\right) \sqrt{\varepsilon}^n$$

$$+\mathbf{U}^{c,j}_{n,L}\left(t, \frac{x-\mu_1^- t}{\sqrt{\varepsilon}}, \ldots, \frac{x-\mu_{N_-}^- t}{\sqrt{\varepsilon}}\right) \sqrt{\varepsilon}^n$$

Actually, depending on the value of j, the ansatz can be written in a simplified manner, but we rather give here a generic ansatz valid for all j. Somewhat related ansatzs can be found in [6] and [7]. The $\underline{\mathbf{U}}^j_{n,L}$ belongs to

$H^\infty\left(\Omega_L^j\right)$. Given that $\mathbf{U}_{n,L}^{*,j}=0$ except for $j=N_-$, we will denote $\mathbf{U}_{n,L}^{*,N_-}$ by $\mathbf{U}_{n,L}^{*}$. The noncharacteristic boundary layer profiles $\mathbf{U}_{n,L,+}^{*}(t,z)$ belongs to $e^{\delta z}H^\infty((0,T)\times\mathbb{R}_-^*)$, for some $\delta>0$. Let us review the characteristic boundary layer profiles $\mathbf{U}_{n,L,+}^{c,j}\left(t,\theta_L^1,\ldots,\theta_L^{N_-}\right)$. For $j=0$, we can use the simplified ansatz $\mathbf{U}_{n,L,+}^{c,0}\left(t,\theta_L^1\right)$ with $\mathbf{U}_{n,L,+}^{c,0}$ belonging to $e^{\delta\theta_L^1}H^\infty((0,T)\times\mathbb{R}_-^*)$, for some $\delta>0$. For $j=N_-$ we can adopt the simplified ansatz $\mathbf{U}_{n,L,+}^{c,N_-}\left(t,\theta_L^{N_-}\right)$ with $\mathbf{U}_{n,L,+}^{c,N_-}$ belonging to $e^{-\delta\theta_L^{N_-}}H^\infty((0,T)\times\mathbb{R}_+^*)$, for some $\delta>0$. For $1\le j\le N_--1$, we have also the simplified ansatz: $\mathbf{U}_{n,L,+}^{c,j}\left(t,\theta_L^j,\theta_L^{j+1}\right)$.
Let us denote by $E_{\mu_j^-}$ the eigenspace of A^- associated to the eigenvalue μ_j^-.
We have then the following decomposition of $\mathbb{R}^N$:

$$\mathbb{R}^N=\bigoplus_{j=1}^{N_-}E_{\mu_j^-}\bigoplus\mathbb{E}_+(A^-),$$

we have thus the associated equality on the projectors:

$$Id=\sum_{j=1}^{N_-}\Pi_j^-+\Pi_{\mathbb{E}_+(A^-)}.$$

$$\mathbf{U}_{n,L,+}^{c,j}\left(t,\theta_L^j,\theta_L^{j+1}\right)=\Pi_j^-\mathbf{U}_{n,L,+}^{c,j}\left(t,\theta_L^j\right)+\Pi_{j+1}^-\mathbf{U}_{n,L,+}^{c,j+1}\left(t,\theta_L^{j+1}\right). \tag{2.4}$$

where $\Pi_j^-\mathbf{U}^{c,j}$ belongs to to $e^{-\delta\theta_L^j}H^\infty((0,T)\times\mathbb{R}_+^*)$, for some $\delta>0$, $\Pi_{j+1}^-\mathbf{U}^{c,j+1}$ belongs to to $e^{\delta\theta_L^{j+1}}H^\infty((0,T)\times\mathbb{R}_-^*)$, for some $\delta>0$. This means that on each subset, after projection, the involved layer profile depends only of one fast characteristic dependent variable.
In a similar way, we have:

$$u_{app,R}^{\varepsilon,j}(t,x)=\sum_{n=0}^{M}\left(\underline{\mathbf{U}}_{n,R}^j(t,x)+\mathbf{U}_{n,R}^{*,j}\left(t,\frac{x}{\varepsilon}\right)\right)\sqrt{\varepsilon}^n$$

$$+\mathbf{U}_{n,R}^{c,j}\left(t,\frac{x-\mu_1^+t}{\sqrt{\varepsilon}},\ldots,\frac{x-\mu_{N_+}^+t}{\sqrt{\varepsilon}}\right)\sqrt{\varepsilon}^n$$

with an ansatz identical to the one exposed before.

Let us explain the different steps of the construction of the approximate solution. We begin by constructing the profiles $(\mathbf{U}_j^*, \underline{\mathbf{U}}_j)$ in cascade, the characteristic profiles $\underline{\mathbf{U}}_j^c$ are then computed as a last step.

Plugging the approximate solution into the equation an identifying the terms with the same power in ε, we obtain our profiles equations. $\left(\mathbf{U}_{R,0}^*, \mathbf{U}_{L,0}^*\right)$ is solution of the following ODE in z:

$$\begin{cases} A^+ \partial_z \mathbf{U}_{R,0}^* - \partial_z^2 \mathbf{U}_{R,0}^* = 0, & \{z > 0\}, \\ A^- \partial_z \mathbf{U}_{L,0}^* - \partial_z^2 \mathbf{U}_{L,0}^* = 0, & \{z < 0\}, \\ \mathbf{U}_{R,0}^*|_{z=0} - \mathbf{U}_{L,0}^*|_{z=0} = -(\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0}), \\ \partial_z \mathbf{U}_{R,0}^*|_{z=0} - \partial_z \mathbf{U}_{L,0}^*|_{z=0} = 0. \end{cases}$$

Since we search for $\mathbf{U}_{R,0}^*$ and $\mathbf{U}_{L,0}^*$ tending towards zero when $z \to \pm\infty$, it is equivalent to solve:

$$\begin{cases} \partial_z \mathbf{U}_{R,0}^* - A^+ \mathbf{U}_{R,0}^* = 0, & \{z > 0\}, \\ \partial_z \mathbf{U}_{L,0}^* - A^- \mathbf{U}_{L,0}^* = 0, & \{z < 0\}, \\ \mathbf{U}_{R,0}^*|_{z=0} - \mathbf{U}_{L,0}^*|_{z=0} = -(\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0}), \\ \partial_z \mathbf{U}_{R,0}^*|_{z=0} - \partial_z \mathbf{U}_{L,0}^*|_{z=0} = 0. \end{cases}$$

Applying $\Pi_{\mathbb{I}}$ to our equations on $\mathbf{U}_{R,0}^*$ and $\mathbf{U}_{L,0}^*$, we get that:
$\Pi_{\mathbb{I}} \mathbf{U}_{R,0}^* = e^{A^+ z} \Pi_{\mathbb{I}} \mathbf{U}_{R,0}^*|_{z=0}$, with

$$\Pi_{\mathbb{I}} \mathbf{U}_{R,0}^*|_{z=0} \in \mathbb{E}_-(A^+) \bigcap \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+) = \{0\},$$

and $\mathbf{U}_{L,0}^* = e^{A^- z} \Pi_{\mathbb{I}} \mathbf{U}_{L,0}^*|_{z=0}$, with

$$\Pi_{\mathbb{I}} \mathbf{U}_{L,0}^*|_{z=0} \in \mathbb{E}_+(A^-) \bigcap \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+) = \{0\}.$$

We obtain then that $\Pi_{\mathbb{I}} \mathbf{U}_{L,0}^* = \Pi_{\mathbb{I}} \mathbf{U}_{R,0}^* = 0$. The same argument apply at any order, giving that, for all $0 \le j \le M$, there holds:

$$\Pi_{\mathbb{I}} \mathbf{U}_{L,j}^* = \Pi_{\mathbb{I}} \mathbf{U}_{R,j}^* = 0.$$

We have just proved that $\mathbf{U}_{R,0}^* = (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \mathbf{U}_{R,0}^*$ and that $\mathbf{U}_{L,0}^* = (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \mathbf{U}_{L,0}^*$. Moreover $\mathbf{U}_{R,0}^* = e^{A^+ z} \mathbf{U}_{R,0}^*|_{z=0}$, with $\mathbf{U}_{R,0}^*|_{z=0} \in \mathbb{E}_-(A^+)$ and $\mathbf{U}_{L,0}^* = e^{A^- z} \mathbf{U}_{L,0}^*|_{z=0}$, with $\mathbf{U}_{L,0}^*|_{z=0} \in \mathbb{E}_+(A^-)$. From the second boundary condition, by using the equation, we get that:

$$A^+ \mathbf{U}_{R,0}^*|_{z=0} = A^- \mathbf{U}_{L,0}^*|_{z=0} \in \mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-),$$

let us denote by σ'_0 this quantity. Returning to the first boundary condition, this leads to:

$$\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0} = -\left((A^+)^{-1} - (A^-)^{-1}\right)\sigma'_0 := \sigma_0,$$

with $\sigma'_0 \in \mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-)$, which gives:

$$\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0} \in \Sigma.$$

For fixed $\sigma_0 \in \Sigma$, the equations giving the profiles $\mathbf{U}^*_{L,0}$ and $\mathbf{U}^*_{R,0}$ are well-posed since we have assumed that $\dim \Sigma = \dim \mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-)$, which is equivalent to $\ker\left((A^+)^{-1} - (A^-)^{-1}\right)\bigcap(\mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-)) = \{0\}$.

We shall now introduce the solution $(\underline{\mathbf{U}}_{L,0}, \underline{\mathbf{U}}_{R,0})$ of the following hyperbolic problem, which is also the limiting hyperbolic problem as ε goes to zero:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,0} + A^- \partial_x \underline{\mathbf{U}}_{L,0} = f^L, & (t,x) \in \Omega_L, \\ \partial_t \underline{\mathbf{U}}_{R,0} + A^+ \partial_x \underline{\mathbf{U}}_{R,0} = f^R, & (t,x) \in \Omega_R, \\ \underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0} \in \Sigma, \\ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \underline{\mathbf{U}}_{L,0}|_{t=0} = h^L, \\ \underline{\mathbf{U}}_{R,0}|_{t=0} = h^R. \end{cases} \tag{2.5}$$

Under our assumptions, this problem is well-posed, as we will prove now. The profiles $\underline{\mathbf{U}}^j_{L,0}$ for $0 \le j \le N_-$ are the restriction of $\underline{\mathbf{U}}_{L,0}$ to Ω^j_L. The same way, the profiles $\underline{\mathbf{U}}^j_{R,0}$ for $0 \le j \le N_+$ are the restriction of $\underline{\mathbf{U}}_{R,0}$ to Ω^j_R.

Proposition 2.12. *If Assumption 2.4 is checked, which means there holds*

$$P^+\mathbb{V}_1 \bigoplus \left(P^+\mathbb{J} + P^-\mathbb{J}\right) \bigoplus P^-\mathbb{V}_2 \bigoplus \Sigma = \mathbb{R}^N,$$

$$\dim \mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-) = \dim \Sigma,$$

and the mapping

$$M := \begin{pmatrix} \Pi_{\mathbb{I}} P^+ (D^+)^{-1} & -\Pi_{\mathbb{I}} P^- (D^-)^{-1} \\ P^+ & -P^- \end{pmatrix}$$

from $\mathbb{J} \times \mathbb{J}$ *into* $\mathbb{I} \times (P^+\mathbb{J} + P^-\mathbb{J})$ *defines an isomorphism between* $\mathbb{J} \times \mathbb{J}$ *and* $\mathbb{I} \times (P^+\mathbb{J} + P^-\mathbb{J})$, *then the transmission problem* (2.5) *has a unique solution.*

Proof. For the sake of simplicity let us denote $u_L := \underline{\mathbf{U}}_{L,0}$ and $u_R := \underline{\mathbf{U}}_{R,0}$. Given our assumptions, there are two nonsingular matrices P^+, P^- and two diagonal matrices D^+ and D^- such that $D^+ = (P^+)^{-1}A^+P^+$ and $D^- = (P^-)^{-1}A^-P^-$. Taking $v_R := (P^+)^{-1}u_R$ and $v_L := (P^-)^{-1}u_L$, we obtain that (v_L, v_R) is solution the equivalent transmission problem:

$$\begin{cases} \partial_t v_R + D^+\partial_x v_R = (P^+)^{-1}f_R, & \{x>0\}, \\ \partial_t v_L + D^-\partial_x v_L = (P^-)^{-1}f_L, & \{x<0\}, \\ P^+v_R|_{x=0} - P^-v_L|_{x=0} \in \Sigma, \\ \partial_x\Pi_{\mathbb{I}}P^+v_R|_{x=0} - \partial_x\Pi_{\mathbb{I}}P^-v_L|_{x=0} = 0, \\ v_L|_{t=0} = (P^-)^{-1}h_L, \\ v_R|_{t=0} = (P^+)^{-1}h_R. \end{cases}$$

Let us denote by $\Pi_{\mathbb{E}_-(D^+)}$ and $\Pi_{\mathbb{E}_+(D^+)}$ the projector associated to the decomposition:

$$\mathbb{R}^N = \mathbb{E}_-(D^+)\bigoplus\mathbb{E}_+(D^+),$$

we define likewise $\Pi_{\mathbb{E}_-(D^-)}$ and $\Pi_{\mathbb{E}_+(D^-)}$. We recall that we have as well the decomposition (2.3). Equation

$$\partial_t v_R + D^+\partial_x v_R = (P^+)^{-1}f_R, \qquad \{x>0\},$$

splits into:

$$v_R = \Pi_{\mathbb{E}_+(D^+)}v_R + \Pi_{\mathbb{E}_-(D^+)}v_R,$$

$$\partial_t(\Pi_{\mathbb{E}_+(D^+)}v_R) + D^+\partial_x(\Pi_{\mathbb{E}_+(D^+)}v_R) = \Pi_{\mathbb{E}_+(D^+)}(P^+)^{-1}f_R, \qquad \{x>0\},$$

and

$$\partial_t(\Pi_{\mathbb{E}_-(D^+)}v_R) + D^+\partial_x(\Pi_{\mathbb{E}_-(D^+)}v_R) = \Pi_{\mathbb{E}_-(D^+)}(P^+)^{-1}f_R, \qquad \{x>0\}.$$

These problems being diagonal, they are tantamount to N scalar, easily solved, independent equations; which shows that: $\Pi_{\mathbb{E}_-(D^+)}v_R$ and $\Pi_{\mathbb{E}_+(D^-)}v_L$ are directly computed from the equation without boundary conditions. Contrary to them, $\Pi_{\mathbb{E}_-(D^+)}v_R$ and $\Pi_{\mathbb{E}_+(D^-)}v_L$ can be computed only when the traces $\Pi_{\mathbb{E}_-(D^+)}v_R|_{x=0}$ and $\Pi_{\mathbb{E}_+(D^-)}v_L|_{x=0}$ are known. The well-posedness of our problem reduces to the algebraic well-posedness of a linear system whose equations are our boundary conditions and the unknowns are the traces $\Pi_{\mathbb{E}_-(D^+)}v_R|_{x=0}$ and $\Pi_{\mathbb{E}_+(D^-)}v_L|_{x=0}$. The boundary condition states that there is $\sigma \in \Sigma$ such that:

(2.6)
$$P^+\Pi_{\mathbb{E}_+(D^+)}v_R - P^-\Pi_{\mathbb{E}_-(D^-)}v_L + \sigma = -P^+\Pi_{\mathbb{E}_-(D^+)}v_R + P^-\Pi_{\mathbb{E}_+(D^-)}v_L.$$

Let us recall a piece of Assumption 2.4:

$$P^+\mathbb{V}_1 \bigoplus \left(P^+\mathbb{J} + P^-\mathbb{J}\right) \bigoplus P^-\mathbb{V}_2 \bigoplus \Sigma = \mathbb{R}^N. \tag{2.7}$$

By (2.7) and since P^+ and P^- are nonsingular, we get the value of the traces on the boundary of:

$$\Pi_1\Pi_{\mathbb{E}_+(D^+)}v_R,$$

$$\Pi_2\Pi_{\mathbb{E}_-(D^-)}v_L,$$

and

$$P^+\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R - P^-\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L,$$

as well as the value of σ. To compute the traces $u_R|_{x=0}$ and $u_L|_{x=0}$, we only lack the knowledge of $\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R|_{x=0}$ and $\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L|_{x=0}$. By the equation, there holds:

$$\partial_x v_R = (D^+)^{-1}\left((P^+)^{-1}f_R - \partial_t v_R\right),$$

$$\partial_x v_L = (D^-)^{-1}\left((P^-)^{-1}f_L - \partial_t v_L\right).$$

The boundary condition $\Pi_{\mathbb{I}}\partial_x v_R|_{x=0} - \Pi_{\mathbb{I}}\partial_x v_L|_{x=0} = 0$ gives then a relation of the form:

$$\Pi_{\mathbb{I}}P^+(D^+)^{-1}\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}\partial_t v_R|_{x=0} - \Pi_{\mathbb{I}}P^-(D^-)^{-1}\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}\partial_t v_L|_{x=0} = q$$

where q is a known continuous function of $t \in (0,T)$, with values polarized on the linear subspace $\mathbb{I}$. Since we have as well

$$P^+\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}\partial_t v_R|_{x=0} - P^-\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}\partial_t v_L|_{x=0} = q'$$

where q' is a known continuous function of $t \in (0,T)$. By Assumption 2.4, for all fixed t there is only one $\partial_t\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R|_{x=0}(t)$ and $\partial_t\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L|_{x=0}(t)$ solution of this linear system of two equations with two unknowns. Moreover, q and q' depending continuously of $t \in (0,T)$, it is also the case for $\partial_t\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R|_{x=0}$ and $\partial_t\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L|_{x=0}$. We have thus:

$$\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R|_{x=0} = \Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}h(0) + \int_0^t \partial_t\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R|_{x=0}(s)\,ds,$$

and

$$\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L|_{x=0} = \Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}h(0) + \int_0^t \partial_t\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L|_{x=0}(s)\,ds,$$

which achieves the computation of the traces $g_L := v_L|_{x=0}$ and $g_R := v_R|_{x=0}$. We obtain then that the hyperbolic problem (2.5), which satisfies nonclassical transmission conditions on the boundary, is actually equivalent to solve two classical well-posed mixed hyperbolic problem with Dirichlet boundary conditions. $u_R = P^+ v_R$, where v_R is solution of:

$$\begin{cases} \partial_t v_R + D^+ \partial_x v_R = (P^+)^{-1} f_R, & \{x > 0\}, \\ v_R|_{x=0} = g_R, \\ v_R|_{t=0} = (P^+)^{-1} h_R. \end{cases}$$

This problem is well-posed because $\Pi_{\mathbb{E}_-(D^+)} g_R$ is incidentally the trace $\Pi_{\mathbb{E}_-(D^+)} v_R|_{x=0}$ computed from the equation without boundary condition. As a consequence, this problem also rewrites:

$$\begin{cases} \partial_t v_R + D^+ \partial_x v_R = (P^+)^{-1} f_R, & \{x > 0\}, \\ \Pi_{\mathbb{E}_+(D^+)} v_R|_{x=0} = \Pi_{\mathbb{E}_+(D^+)} g_R. \\ v_R|_{t=0} = (P^+)^{-1} h_R. \end{cases}$$

which is a mixed hyperbolic problem satisfying a Uniform Lopatinski condition. The same way v_L is the solution of the following mixed hyperbolic problem satisfying a Uniform Lopatinski condition:

$$\begin{cases} \partial_t v_L + D^- \partial_x v_L = (P^-)^{-1} f_L, & \{x < 0\}, \\ \Pi_{\mathbb{E}_-(D^-)} v_L|_{x=0} = \Pi_{\mathbb{E}_-(D^-)} g_L. \\ v_L|_{t=0} = (P^-)^{-1} h_L, \end{cases}$$

and u_L is obtained by: $u_L = P^+ v_L$, which shows that problem (2.5) is well-posed.

□

Proof of the well-posedness of the transmission problem (2.5) **under Assumption 2.6**

There holds:

(2.8)

$$\begin{cases} \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} + \Pi_{\mathbb{I}} A^- \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} = \Pi_{\mathbb{I}} f^L - \Pi_{\mathbb{I}} A^- \partial_x (\Pi_{\mathbb{V}} + \Pi_\Sigma) \underline{\mathbf{U}}_{L,0}, & \{x < 0\}. \\ \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} + \Pi_{\mathbb{I}} A^+ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} = \Pi_{\mathbb{I}} f^R - \Pi_{\mathbb{I}} A^+ \partial_x (\Pi_{\mathbb{V}} + \Pi_\Sigma) \underline{\mathbf{U}}_{R,0}, & \{x > 0\}. \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{t=0} = \Pi_{\mathbb{I}} h^L, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{t=0} = \Pi_{\mathbb{I}} h^R. \end{cases}$$

Hence, by Assumption 2.6, we have:

(2.9)
$$\begin{cases} \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} + A^- \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} = \Pi_{\mathbb{I}} f^L - \Pi_{\mathbb{I}} A^- \partial_x (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \underline{\mathbf{U}}_{L,0}, \quad \{x < 0\}. \\ \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} + A^+ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} = \Pi_{\mathbb{I}} f^R - \Pi_{\mathbb{I}} A^+ \partial_x (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \underline{\mathbf{U}}_{R,0}, \quad \{x > 0\}. \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{t=0} = \Pi_{\mathbb{I}} h^L, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{t=0} = \Pi_{\mathbb{I}} h^R. \end{cases}$$

Let us now introduce $\underline{\mathbf{V}}_{L,0} = (Id - \Pi_{\mathbb{I}})\underline{\mathbf{U}}_{L,0}$, $\underline{\mathbf{V}}_{R,0} = (Id - \Pi_{\mathbb{I}})\underline{\mathbf{U}}_{R,0}$, applying then $(Id - \Pi_{\mathbb{I}})$ to our equation, we get the following:

$$\begin{cases} \partial_t \underline{\mathbf{V}}_{L,0} + (Id - \Pi_{\mathbb{I}}) M^- \partial_x \underline{\mathbf{V}}_{L,0} = (Id - \Pi_{\mathbb{I}}) f^L, \qquad \{x < 0\}. \\ \partial_t \underline{\mathbf{V}}_{R,0} + (Id - \Pi_{\mathbb{I}}) M^+ \partial_x \underline{\mathbf{V}}_{R,0} = (Id - \Pi_{\mathbb{I}}) f^R, \qquad \{x > 0\}. \\ \underline{\mathbf{V}}_{R,0}|_{x=0} - \underline{\mathbf{V}}_{L,0}|_{x=0} \in \Sigma, \\ \underline{\mathbf{V}}_{L,0}|_{t=0} = (Id - \Pi_{\mathbb{I}}) h^L, \\ \underline{\mathbf{V}}_{R,0}|_{t=0} = (Id - \Pi_{\mathbb{I}}) h^R. \end{cases}$$

Referring the reader to the analysis performed in the multi-D case treated in [7] for further details, this mixed hyperbolic problem is well-posed provided that it satisfies the Uniform Lopatinski Condition stating that

$$\mathbb{E}_-((Id - \Pi_{\mathbb{I}})M^-) \bigoplus \mathbb{E}_+((Id - \Pi_{\mathbb{I}})M^+) \bigoplus \Sigma = \mathbb{V} \bigoplus \Sigma.$$

As we will see, we can now compute the solution of (2.9). Indeed there is an unique

$$\mathbf{g}(t) := \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} = \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0},$$

which depends continuously of $t \in (0, T)$, satisfying our boundary conditions provided that

$$\mathrm{Ker}((A^+)^{-1} - (A^-)^{-1}) \bigcap \mathbb{I} = \{0\}.$$

Indeed, by using the equation, we get that $\partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0$ writes as well:

$$\left((A^+)^{-1} - (A^-)^{-1}\right) \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} = q'',$$

where q'' stands for a known function continuous in t. As a result, we obtain that:

$$\Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0}(t) = \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0}(t) = \Pi_{\mathbb{I}} h(0) + \int_0^t \mathbf{g}(s)\, ds,$$

which proves the well-posedness of the hyperbolic problem (2.5) under Assumption 2.6.

Since Assumption 2.4 being checked is a sufficient but also necessary condition in order for problem (2.5) to be well-posed, we get then that:

$$[\text{Assumption 2.6} \Rightarrow \text{Assumption 2.4}].$$

Since the problem (2.5) is well-posed, $u_L|_{x=0} - u_R|_{x=0} := \sigma_0 \in \Sigma$ is known and thus $\mathbf{U}^*_{L,0}$ and $\mathbf{U}^*_{R,0}$ as well. This scheme of construction can be carried out at any order. Let us show how the the other profiles are constructed:

$$\begin{cases} A^+\partial_z \mathbf{U}^*_{R,1} - \partial_z^2 \mathbf{U}^*_{R,1} = 0, & \{z > 0\}, \\ A^-\partial_z \mathbf{U}^*_{L,1} - \partial_z^2 \mathbf{U}^*_{L,1} = 0, & \{z < 0\}, \\ \mathbf{U}^*_{R,1}|_{z=0} - \mathbf{U}^*_{L,1}|_{x=0} = -(\underline{\mathbf{U}}_{R,1}|_{x=0} - \underline{\mathbf{U}}_{L,1}|_{x=0}), \\ \partial_z \mathbf{U}^*_{R,1}|_{z=0} - \partial_z \mathbf{U}^*_{L,1}|_{z=0} = 0. \end{cases}$$

$$\begin{cases} A^+\partial_z \mathbf{U}^*_{R,2} - \partial_z^2 \mathbf{U}^*_{R,2} = -\partial_t \mathbf{U}^*_{R,0}, & \{z > 0\}, \\ A^-\partial_z \mathbf{U}^*_{L,2} - \partial_z^2 \mathbf{U}^*_{L,2} = -\partial_t \mathbf{U}^*_{L,0}, & \{z < 0\}, \\ \mathbf{U}^*_{R,2}|_{z=0} - \mathbf{U}^*_{L,2}|_{x=0} = -(\underline{\mathbf{U}}_{R,2}|_{x=0} - \underline{\mathbf{U}}_{L,2}|_{x=0}), \\ \partial_z \mathbf{U}^*_{R,2}|_{z=0} - \partial_z \mathbf{U}^*_{L,2}|_{z=0} = -\left(\partial_x \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,0}|_{x=0}\right). \end{cases}$$

$\Pi_2 \mathbf{U}^*_{L,2} = \Pi_2 \mathbf{U}^*_{R,2} = 0$, which does not contradict our previous computations since $\mathbf{\Pi}_2\left(\partial_x \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,0}|_{x=0}\right) = 0$. Actually for $n \geq 2$, we have:

$$\begin{cases} A^+\partial_z \mathbf{U}^*_{R,n} - \partial_z^2 \mathbf{U}^*_{R,n} = -\partial_t \mathbf{U}^*_{R,n-2}, & \{z > 0\}, \\ A^-\partial_z \mathbf{U}^*_{L,n} - \partial_z^2 \mathbf{U}^*_{L,n} = -\partial_t \mathbf{U}^*_{L,n-2}, & \{z < 0\}, \\ \mathbf{U}^*_{R,n}|_{z=0} - \mathbf{U}^*_{L,n}|_{x=0} = -(\underline{\mathbf{U}}_{R,n}|_{x=0} - \underline{\mathbf{U}}_{L,n}|_{x=0}), \\ \partial_z \mathbf{U}^*_{R,n}|_{z=0} - \partial_z \mathbf{U}^*_{L,n}|_{z=0} = -\left(\partial_x \underline{\mathbf{U}}_{R,n-2}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,n-2}|_{x=0}\right). \end{cases}$$

$(\underline{\mathbf{U}}_{L,n}, \underline{\mathbf{U}}_{R,n})$ are given by:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,n} + A^-\partial_x \underline{\mathbf{U}}_{L,n} = \partial_x^2 \underline{\mathbf{U}}_{L,n-2}, & \{x < 0\}. \\ \partial_t \underline{\mathbf{U}}_{R,n} + A^+\partial_x \underline{\mathbf{U}}_{R,n} = \partial_x^2 \underline{\mathbf{U}}_{R,n-2}, & \{x > 0\}. \\ \underline{\mathbf{U}}_{R,n}|_{x=0} - \underline{\mathbf{U}}_{L,n}|_{x=0} \in p_n + \Sigma, \\ \partial_x \Pi_2 \underline{\mathbf{U}}_{R,n}|_{x=0} - \partial_x \Pi_2 \underline{\mathbf{U}}_{L,n}|_{x=0} = 0, \\ \underline{\mathbf{U}}_{L,n}|_{t=0} = 0, \\ \underline{\mathbf{U}}_{R,n}|_{t=0} = 0. \end{cases} \tag{2.10}$$

where p_n is computed using the equations on $\mathbf{U}^*_{R,n}$ and $\mathbf{U}^*_{L,n}$. This mixed hyperbolic problem is well-posed for the same reasons as the mixed hyperbolic problems giving $(\underline{\mathbf{U}}_{L,0}, \underline{\mathbf{U}}_{R,0})$. The profiles $\underline{\mathbf{U}}^j_{L,n}$ for $0 \leq j \leq N_-$ are the restriction of $\underline{\mathbf{U}}_{L,n}$ to Ω^j_L. The same way, the profiles $\underline{\mathbf{U}}^j_{R,n}$ for $0 \leq j \leq N_+$ are the restriction of $\underline{\mathbf{U}}_{R,n}$ to Ω^j_R.

Referring to (2.4), we have actually to compute the profiles $\Pi^-_j \mathbf{U}^{c,j}_{n,L,\pm}(t, \theta^j_L)$ and $\Pi^+_j \mathbf{U}^{c,j}_{n,R,\pm}(t, \theta^j_R)$. Since the profiles equations satisfied by $\Pi^-_j \mathbf{U}^{c,j}_{n,L,\pm}$ and $\Pi^+_j \mathbf{U}^{c,j}_{n,R,\pm}$ are of the same form, we will only focus on the computation of the profiles $\mathbf{U}^{c,\pm}_{L,n}(t, z_j) := \Pi^-_j \mathbf{U}^{c,j}_{n,L,\pm}(t, \theta^j_L)$ for some j. Observe that, the pieces of solutions $(\underline{\mathbf{U}}_{L,j}, \underline{\mathbf{U}}_{R,j})$ glued together compose in general a function belonging to $C^0((0,T) \times \mathbb{R})$ but not to $C^1((0,T) \times \mathbb{R})$. Since the characteristic profiles allow the glued together approximate solution to belong to $C^1((0,T) \times \mathbb{R})$, computing the characteristics layer profiles amounts to solve equations of the form:

$$\begin{cases} \partial_t \mathbf{U}^{c,+}_{L,n} - \partial^2_{z_j} \mathbf{U}^{c,+}_{L,n} = 0, \qquad \{z_j > 0\}, \\ \partial_t \mathbf{U}^{c,-}_{L,n} - \partial^2_{z_j} \mathbf{U}^{c,-}_{L,n} = 0, \qquad \{z_j < 0\}, \\ [\mathbf{U}^c_{L,n}]_j(t) = -[\underline{\mathbf{U}}_{L,n}]_{\Gamma_j}(t), \quad \forall t \in (0,T), \\ [\partial_x \mathbf{U}^c_{L,n}]_j(t) = -\dfrac{1}{2}\left([\partial_x \underline{\mathbf{U}}_{L,n-1}]_{\Gamma_j}(t) + [\partial_x U^c_{L,n-1}]_j(t)\right), \quad \forall t \in (0,T), \\ \mathbf{U}^{c+}_{L,n}|_{t=0} = 0, \\ \mathbf{U}^{c-}_{L,n}|_{t=0} = 0, \end{cases}$$

where $[\omega]_j(t) = \lim_{z_j \to 0^+} \omega(t, z_j) - \lim_{z_j \to 0^-} \omega(t, z_j)$ and $[\omega']_{\Gamma_j}(t) = \lim_{x \to \mu^-_j t, x > \mu^-_j t} \omega'(t,x) - \lim_{x \to \mu^-_j t, x < \mu^-_j t} \omega'(t,x)$. These profiles equations are clearly well-posed, using the same argument used in [7]. To sum up, we have constructed $u^\varepsilon_{app} := u^\varepsilon_{R,app} \mathbf{1}_{x \geq 0} + u^\varepsilon_{L,app} \mathbf{1}_{x<0}$ such that:

$$\begin{cases} \partial_t u^\varepsilon_{app} + A(x) \partial_x u^\varepsilon_{app} - \varepsilon \partial^2_x u^\varepsilon_{app} = f + \varepsilon^M R^\varepsilon, \quad (t,x) \in \Omega, \\ u^\varepsilon_{app}|_{t=0} = h. \end{cases}$$

2.2 Stability estimates.

This time, we will rather note

$$u^\varepsilon_{app} := u^{\varepsilon+}_{app}(t,x) \mathbf{1}_{x>0} + u^{\varepsilon-}_{app}(t,-x) \mathbf{1}_{x<0}.$$

By linearity, the error equation writes, for $w^\varepsilon = u^\varepsilon_{app} - u^\varepsilon$:

$$\begin{cases} \partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon\partial_x^2 w^\varepsilon = \varepsilon^M R^\varepsilon, & (t,x) \in \Omega, \\ w^\varepsilon|_{t=0} = 0. \end{cases}$$

Since our method of estimation comes from pseudodifferential calculus, we have to perform a tangential Fourier-Laplace transform of the problem. For this purpose, it is necessary to extend the definition of our error, in order for it to be defined for all time $t \in \mathbb{R}$. We first perform an extension of w^ε to $\{t < 0\}$ as follows: $\tilde{w}^\varepsilon := \begin{cases} w^\varepsilon \ on \ (0,T) \\ 0 \ on \ t < 0 \end{cases}$ but, for fixed positive ε, $w^\varepsilon \in C((0,T) : L^2(\mathbb{R}))$ and $w^\varepsilon|_{t=0} = 0$ thus $\tilde{w}^\varepsilon$ belongs to $C((-\infty, T] : L^2(\mathbb{R}))$. Moreover, $\partial_t \tilde{w}^\varepsilon$ has no Dirac measure on $\{t = 0\}$ and thus $\tilde{w}^\varepsilon$ is solution of:

$$\partial_t \tilde{w}^\varepsilon + A(x)\partial_x \tilde{w}^\varepsilon - \varepsilon\partial_x^2 \tilde{w}^\varepsilon = \varepsilon^M \tilde{R}^\varepsilon, \qquad (t,x) \in (-\infty, T] \times \mathbb{R},.$$

where $\tilde{R}^\varepsilon := \begin{cases} R^\varepsilon & if & t \in (0,T), \\ 0 & on & t < 0. \end{cases}$
Finally, we denote by $\underline{\tilde{R}}^\varepsilon$, $\tilde{R}^\varepsilon$ extended by 0 outside $(0,T) \times \mathbb{R}$. Let us now proceed with the extension of our error to $t > T$. We call by $\underline{\tilde{w}}^\varepsilon$ the unique solution of:

$$\begin{cases} \mathcal{H}\underline{\tilde{w}}^\varepsilon - \varepsilon\partial_x^2 \underline{\tilde{w}}^\varepsilon = \varepsilon^M \underline{\tilde{R}}^\varepsilon, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \underline{\tilde{w}}^\varepsilon|_{t<0} = 0. \end{cases} \tag{2.11}$$

Note well that the restriction of $\underline{\tilde{w}}^\varepsilon$ to Ω is w^ε. For the sake of simplicity, we will still denote $\underline{\tilde{w}}^\varepsilon$ [resp $\underline{\tilde{R}}^\varepsilon$] by w^ε [resp R^ε] in what follows.

To begin with, let us rewrite the problem (2.11) in a convenient form. w^ε is solution of:

$$\partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon\partial_x^2 w^\varepsilon = \varepsilon^M R^\varepsilon, \qquad (t,x) \in \mathbb{R} \times \mathbb{R},$$

We denote then by $\hat{w}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} w^{\varepsilon\pm})$ and $\hat{R}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} R^{\varepsilon\pm})$, where $\mathcal{F}$ stands for the tangential Fourier transform (with respect to t) and the $\pm$ superscripts indicates restrictions to $\{\pm x > 0\}$, we have then:

$$\begin{cases} (i\tau + \gamma)\hat{w}^{\varepsilon+} + A^+\partial_x \hat{w}^{\varepsilon+} - \varepsilon\partial_x^2 \hat{w}^{\varepsilon+} = \varepsilon^M \hat{R}^{\varepsilon+}, & \{x > 0\}, \\ (i\tau + \gamma)\hat{w}^{\varepsilon-} + A^-\partial_x \hat{w}^{\varepsilon-} - \varepsilon\partial_x^2 \hat{w}^{\varepsilon-} = \varepsilon^M \hat{R}^{\varepsilon-}, & \{x < 0\}, \\ \hat{w}^{\varepsilon+}|_{x=0} - \hat{w}^{\varepsilon-}|_{x=0} = 0, \\ \partial_x \hat{w}^{\varepsilon+}|_{x=0} - \partial_x \hat{w}^{\varepsilon-}|_{x=0} = 0. \end{cases} \tag{2.12}$$

Remark that, by taking γ big enough, the restrictions of the solution w^ε of (2.11) to $\{\pm x > 0\}$ are given by:

$$w^{\varepsilon\pm} = e^{\gamma t}\mathcal{F}^{-1}(\hat{w}^{\varepsilon\pm}),$$

where $(\hat{w}^{\varepsilon+}, \hat{w}^{\varepsilon-})$ are the solutions of the transmission problem (2.12).

Taking $W^{\varepsilon\pm}(i\tau+\gamma, x) = \begin{pmatrix} \hat{w}^{\varepsilon\pm} \\ \\ \varepsilon\partial_x \hat{w}^{\varepsilon\pm} \end{pmatrix}$,

$$\begin{cases} \partial_x W^{\varepsilon+} = \begin{pmatrix} \partial_x \hat{w}^{\varepsilon+} \\ \\ \varepsilon\partial_x^2 \hat{w}^{\varepsilon+} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} Id \\ \\ (i\tau+\gamma) & \frac{1}{\varepsilon}A^+ \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon+} \\ \\ \varepsilon\partial_x \hat{w}^{\varepsilon+} \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \varepsilon^M \hat{R}^{\varepsilon+} \end{pmatrix}, \\ \\ \partial_x W^{\varepsilon-} = \begin{pmatrix} \partial_x \hat{w}^{\varepsilon-} \\ \\ \varepsilon\partial_x^2 \hat{w}^{\varepsilon-} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} Id \\ \\ (i\tau+\gamma) & \frac{1}{\varepsilon}A^- \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon-} \\ \\ \varepsilon\partial_x \hat{w}^{\varepsilon-} \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \varepsilon^M \hat{R}^{\varepsilon-} \end{pmatrix}, \\ W^{\varepsilon+}|_{x=0} - W^{\varepsilon-}|_{x=0} = 0. \end{cases}$$

We note $\zeta = (\tau, \gamma)$ and $\tilde{\zeta} = (\varepsilon\tau, \varepsilon\gamma)$. Multiplying the previous equation by ε gives:

$$\begin{cases} \partial_z W^{\varepsilon+} - \mathbb{A}^+(\tilde{\zeta})W^{\varepsilon+} = G^+, & \{z > 0\}, \\ \partial_z W^{\varepsilon-} - \mathbb{A}^-(\tilde{\zeta})W^{\varepsilon-} = \tilde{G}^-, & \{z < 0\}, \\ W^{\varepsilon+}|_{z=0} = W^{\varepsilon-}|_{z=0}, \end{cases} \tag{2.13}$$

where

$$G^\pm = \begin{pmatrix} 0 \\ \varepsilon^{M+1}\hat{R}^{\varepsilon\pm} \end{pmatrix},$$

and z stands for the fast variable $\frac{x}{\varepsilon}$. From this point onwards, since nothing differs from the proof of stability by symmetrizers done in [7], we give the result:

Proposition 2.13. *There is $C > 0$ such that for all $0 < \varepsilon < 1$, there holds:*

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^2(\Omega)} \leq C\varepsilon^{M-1}.$$

2.3 The main result.

We recall that u^ε stands for the solution of the viscous problem (1.2) and $u := u^+\mathbf{1}_{x\geq 0} + u^-\mathbf{1}_{x<0}$, where (u^+, u^-) is solution of the well-posed transmission problem (1.3) or (2.5).

Theorem 2.14. *u^ε converges towards u in $L^2(\Omega)$ as ε tends to zero. More precisely, there is $C > 0$, independent of ε such that:*

$$\|u^\varepsilon - u\|_{L^2((0,T)\times\mathbb{R})} \leq C\varepsilon.$$

Proof. By construction of our approximate solution u^ε_{app}, we have:

$$\|u^\varepsilon - u\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon).$$

Hence, by constructing our approximate solution at a sufficient order M, Proposition 2.13 ends the proof. □

3 Stability study for 2×2 nonconservative systems.

In this chapter, our goal is to analyze the uniform Evans condition for 2×2 systems. We limit ourselves to this framework due to the fast increasing complexity of the computations with the size of the systems. This analysis is not trivial to perform, as witness, even for 2×2 systems, a sufficient and necessary reformulation of the Evans Condition, not involving any frequencies, has yet to be found out. Our point here is to give a brief overview of the link existing between the matrices A^- and A^+ and the uniform Evans condition being checked. As a result of our study, the uniform Evans Condition does not appear as a very restrictive assumption, but, on the other hand, is not always satisfied. The uniform Evans Condition writes as the nonvanishing of an Evans function for a given range of frequencies. This Evans function is a determinant that can be written in several equivalent ways. D and $\tilde{D}$ are two equivalent Evans functions iff, for all $\zeta \neq 0$,

$$D(\zeta) = 0 \Leftrightarrow \tilde{D}(\zeta) = 0.$$

We will begin by giving the expression of an Evans function for medium frequencies, then we will introduce asymptotic Evans functions for $|\zeta| \to \infty$ (high frequencies) and $|\zeta| \to 0^+$ (low frequencies). Our results for 2×2 systems are divided the same way. The study of the low frequency behavior is the more technical, since some arguments break down due to eigenvalues crossing the imaginary axis. The specific analysis for low frequencies involves the continuous extension of some linear subspaces intervening in the formulation of the Evans function. A part of our analysis is devoted to the computation of these extensions for some 2×2 systems. During our study, we achieve the proof of Proposition (2.10).

3.1 Spectral analysis of the symbol $\mathbb{A}^{\pm}$.

The expression of an Evans function relies on the computation of the linear subspaces $\mathbb{E}_{-}(\mathbb{A}^{+})$ and $\mathbb{E}_{+}(\mathbb{A}^{-})$. An important point is that, except for low frequencies, the eigenvalues of $\mathbb{A}^{+}$ and $\mathbb{A}^{-}$ do not cross the imaginary axis. $\mathbb{A}^{+}$ and $\mathbb{A}^{-}$ have both N eigenvalues with positive real part and N eigenvalues with negative real part. As a consequence, if the Evans condition holds, for all ζ in an open subset not containing $\{0\}$, there holds: $\mathbb{E}_{-}(\mathbb{A}^{+}) \bigoplus \mathbb{E}_{+}(\mathbb{A}^{-}) = \mathbb{C}^{2N}$. We will now show that the eigenvectors of $\mathbb{A}^{\pm}$ can be deduced from the eigenvectors of $A^{\pm}$. Denote by v_i^{+} [resp v_i^{-}]the normalized eigenvector associated to the eigenvalue λ_i^{+} of A^{+} [resp λ_i^{-} of A^{-}]. The eigenvectors of $\mathbb{A}^{+}$ associated to the eigenvalues with negative real parts, denoted by $(\mu_i^{+})_{1 \leq i \leq N}$, are given by:

$$\left(\mathbf{w}_i^{+}\right)_{1 \leq i \leq N} := \left(\begin{array}{c} v_i^{+} \\ \mu_i^{+} v_i^{+} \end{array}\right)_{1 \leq i \leq N}.$$

Likewise, the eigenvectors of $\mathbb{A}^{+}$ associated to the eigenvalues with positive real parts, noted $(\mu_i^{+})_{N+1 \leq i \leq 2N}$, are given by:

$$\left(\mathbf{w}_i^{+}\right)_{N+1 \leq i \leq 2N} := \left(\begin{array}{c} v_i^{+} \\ \mu_{N+i}^{+} v_i^{+} \end{array}\right)_{N+1 \leq i \leq 2N}.$$

The family $\left(\mathbf{w}_i^{+}\right)_{1 \leq i \leq N}$ is a basis of $\mathbb{E}_{-}(\mathbb{A}^{+})$. Moreover, μ_i^{+} satisfy:

$$\mu_i^{+2} - \lambda_j^{+} \mu_i^{+} - (i\tau + \gamma) = 0.$$

Proof. Denote μ an eigenvalue of $\mathbb{A}^{+}$ and $\mathbf{v} = \left(\begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \end{array}\right)$ an eigenvector associated to μ.

$$\begin{cases} \mathbf{v}_2 = \mu \mathbf{v}_1 \\ A^{+} \mathbf{v}_1 = \dfrac{\mu^2 - (i\tau + \gamma)}{\mu} \mathbf{v}_1 \end{cases}$$

Since $\mathbf{v}_1 = 0_{\mathbb{R}^N} \Rightarrow \mathbf{v} = 0_{\mathbb{C}^{2N}}$, $\mathbf{v}_1$ is an eigenvector of A^{+} associated to the eigenvalue $\frac{\mu^2 - (i\tau+\gamma)}{\mu}$. Hence there is $1 \leq j \leq N$ such that $\lambda_j^{+} = \frac{\mu^2 - (i\tau+\gamma)}{\mu}$.

We will show here that, for all $(\tau,\gamma) \neq 0$, the eigenvalues of $\mathbb{A}^+$ are all semi-simple and that N of them have positive real part and N of them have negative real part. This result is deduced from the fact that we can associate to each eigenvalues of A^+ two eigenvalues of $\mathbb{A}^+$: one with positive real part and one with negative real part. Moreover, for each eigenvalue of $\mathbb{A}^+$ the associated eigenvector can be directly constructed by using the eigenvector associated to the corresponding eigenvalue of A^+ as stated above. The eigenvalues of $\mathbb{A}^+$ are the roots of P defined by:

$$P(\mu) = \mu^2 - \lambda\mu - (i\tau + \gamma).$$

Note that the roots of P^+ are:

$$\mu_- = \frac{1}{2}\left(\lambda - sign\left(cos(\theta^+/2)\right)\sqrt{r^+}e^{i(\theta^+/2)}\right),$$

$$\mu_+ = \frac{1}{2}\left(\lambda + sign\left(cos(\theta^+/2)\right)\sqrt{r^+}e^{i(\theta^+/2)}\right).$$

where $r^+ = \sqrt{(\lambda^2+4\gamma)^2+16\tau^2}$ and $\theta^+ = \arctan\frac{4\tau}{\lambda^2+4\gamma}$. The $\pm$ subscripts in the right above notations relates to the sign of the real part of the concerned eigenvalues. There holds:

$$sign\left(sin(\theta^+/2)\right) = sign(\tau) \times sign\left(cos(\theta^+/2)\right).$$

We deduce from it that:

$$\mu_- = \frac{1}{2}\lambda - \frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}+1\right)$$

$$-i\ sign(\tau)\frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}\right)$$

and

$$\mu_+ = \frac{1}{2}\lambda + \frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}+1\right)$$

$$+i\ sign(\tau)\frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}\right)$$

Notice that we have:

$$\mu_+|_{(\tau,\gamma)=(0,0)} = \lambda$$

Taking into account that, due to the noncharacteristic boundary assumption, $\lambda \neq 0$, there are two constants C_1 and C_2 such that, for all $\tau \in \mathbb{R}$ and $\gamma > 0$, there holds:

$$\Re e(\mu_+) > C_1 > 0, \qquad \Re e(\mu_-) < C_2 < 0.$$

Indeed, studying the sign of $\Re e(\mu_+^+)$ and $\Re e(\mu_-^-)$ all amounts to the study of the sign of the following expression:

$$2\lambda((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - sign(\lambda)\left(\lambda^2+4\gamma+((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right),$$

which has the same sign as:

$$sign(\lambda)\left(4\lambda^2((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}} - \left(\lambda^2+4\gamma+((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right)^2\right)$$

$$= -sign(\lambda)\left((\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right)$$

Using that $\gamma \geq 0$, we have:

$$(\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}$$

$$\geq (\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(-8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}$$

Noticing that

$$(\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(-8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}} = (\lambda^2+4\gamma-((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}})^2 \geq 0,$$

with the equality only holding for $(\tau,\gamma)=0$, it gives that, if $(\tau,\gamma)\neq(0,0)$:

$$sign(2\lambda((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - sign(\lambda)\left(\lambda^2+4\gamma+((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right)) = -sign(\lambda)$$

Hence we have:

- If $\lambda < 0$, then $\Re e(\mu_+) \geq 0$, with the equality holding only for $(\tau,\gamma)=0$. Moreover $\Re e(\mu_-) < 0$ for all $(\tau,\gamma) \in \mathbb{R}\times\mathbb{R}^+$.
- If $\lambda > 0$ then $\Re e(\mu_+) > 0$ for all $(\tau,\gamma)\in\mathbb{R}\times\mathbb{R}^+$. In addition, $\Re e(\mu_-) \leq 0$, with the equality holding only for $(\tau,\gamma)=0$.

□

The same way, the eigenvectors of $\mathbb{A}^-$ associated to the eigenvalues with positive real parts denoted by $(\mu_i^-)_{1\leq i\leq N}$ are given by:

$$\left(\mathbf{w}_i^-\right)_{1\leq i\leq N} := \begin{pmatrix} v_i^- \\ \mu_i^- v_i^- \end{pmatrix}_{1\leq i\leq N}.$$

The eigenvectors of $\mathbb{A}^-$ associated to the eigenvalues with negative real parts denoted by $(\mu_i^-)_{N+1\leq i\leq 2N}$ are given by:

$$\left(\mathbf{w}_i^-\right)_{N+1\leq i\leq 2N} := \begin{pmatrix} v_i^- \\ \mu_{N+i}^- v_i^- \end{pmatrix}_{N+1\leq i\leq 2N}.$$

The family $\left(\mathbf{w}_i^+\right)_{1\leq i\leq N}$ is a basis of $\mathbb{E}_+(\mathbb{A}^-)$. Moreover the $\mu_i^-(\tau,\gamma)$ satisfy:

$$\lambda_j^- = \mu_i^-(\tau,\gamma) - \frac{i\tau+\gamma}{\mu_i^-(\tau,\gamma)}.$$

3.2 Expression of an Evans function.

For medium frequencies, that is to say for ζ belonging to a bounded open subset of $\mathbb{R}\times\mathbb{R}^+$ not containing 0, an Evans function is given by:

$$D(\zeta) := \begin{vmatrix} v_1^+ & \dots & v_N^+ & v_1^- & \dots & v_N^- \\ \mu_1^+(\zeta)v_1^+ & \dots & \mu_N^+(\zeta)v_N^+ & \mu_1^-(\zeta)v_1^- & \dots & \mu_N^-(\zeta)v_N^- \end{vmatrix}.$$

For the asymptotic Evans function, when $|\zeta|\to\infty$, we take:

$$\tilde{D}(\zeta) := \begin{vmatrix} v_1^+ & \dots & v_N^+ & v_1^- & \dots & v_N^- \\ \frac{\mu_1^+(\zeta)}{\Lambda(\zeta)}v_1^+ & \dots & \frac{\mu_N^+(\zeta)}{\Lambda(\zeta)}v_N^+ & \frac{\mu_1^-(\zeta)}{\Lambda(\zeta)}(\hat{\zeta})v_1^- & \dots & \frac{\mu_N^-(\zeta)}{\Lambda(\zeta)}v_N^- \end{vmatrix},$$

Due to its specificity, the asymptotic Evans function for low frequencies will be introduced in the section right below, along with the needed material.

3.3 Introduction to a low frequency Evans function.

We will now perform here a detailed analysis of the Evans function for low frequencies. Since some eigenvalues, that we will call hyperbolic, of $\mathbb{A}^\pm$ vanishes for $\tilde{\zeta} = 0$, the associated positive or negative space of $\mathbb{A}^\pm$ cease to be well-defined for low frequencies. Although it is the case, we will show we can extend the definition of those spaces in a continuous way. We will later provide explicit computations of those limiting spaces in section 3.7. The associated asymptotic Evans function will be computed during section 3.8, its nonvanishing meaning that the uniform Evans Condition becomes equivalent to the Evans Condition. The main idea behind our proof is that only the hyperbolic eigenvalues and the associated eigenvectors have to be recomputed for low frequencies. In a first step, we will introduce the appropriate scaling for the low frequency analysis of what corresponds to the hyperbolic block. We recall that $\mathbb{A}^{\pm}$ denotes the following 4×4 sized matrix:

$$\mathbb{A}^{\pm}(\tilde{\zeta}) := \begin{pmatrix} 0 & Id \\ (i\tilde{\tau}+\tilde{\gamma})Id & A^{\pm} \end{pmatrix},$$

Moreover, it intervenes in an ODE of the form:

$$\partial_z \begin{pmatrix} w^{\pm} \\ \partial_z w^{\pm} \end{pmatrix} = \mathbb{A}^{\pm}(\tilde{\zeta}) \begin{pmatrix} w^{\pm} \\ \partial_z w^{\pm} \end{pmatrix} + F^{\pm},$$

We have then:

$$\partial_z \begin{pmatrix} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{pmatrix} := \begin{pmatrix} 0 & \rho Id \\ \rho^{-1}(i\tilde{\tau}+\tilde{\gamma})Id & A^{\pm} \end{pmatrix} \begin{pmatrix} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{pmatrix} := \rho\check{\mathbb{A}}(\check{\zeta},\rho) \begin{pmatrix} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{pmatrix},$$

where

$$\check{\mathbb{A}}^{\pm}(\check{\zeta},\rho) := \begin{pmatrix} 0 & Id \\ \rho^{-1}(i\check{\tau}+\check{\gamma})Id & \rho^{-1}A^{\pm} \end{pmatrix}$$

with $\check{\tau} := \frac{\tilde{\tau}}{\rho}$ and $\check{\gamma} := \frac{\tilde{\gamma}}{\rho}$.

For $\tilde{\gamma} > 0$,

$$\mathbb{E}_{-}(\mathbb{A}^{+}) = \mathbb{E}_{-}^{H}(\mathbb{A}^{+}) \bigoplus \mathbb{E}_{-}^{P}(\mathbb{A}^{+}),$$

where $\mathbb{E}_{-}^{H}(\mathbb{A}^{+})$ is the space generated by the generalized eigenvectors of $\mathbb{A}^{+}$ associated to the the hyperbolic eigenvalues of $\mathbb{A}^{+}$ with negative real part. The same way, $\mathbb{E}_{-}^{P}(\mathbb{A}^{+})$ stands for the space generated by the generalized eigenvectors of $\mathbb{A}^{+}$ associated to the the parabolic eigenvalues of $\mathbb{A}^{+}$ with negative real part. By opposition to the hyperbolic eigenvalues, the parabolic eigenvalues does not cross the imaginary axis even for $\tilde{\zeta} = 0$. Remark that the dimensions of $\mathbb{E}_{-}^{H}(\mathbb{A}^{+})$ and $\mathbb{E}_{-}^{P}(\mathbb{A}^{+})$ are constant. Viewing temporarily $\check{\zeta}$ as a parameter, we introduce the following decomposition:

$$\mathbb{E}_{-}(\check{\mathbb{A}}^{+}) = \mathbb{E}_{-}^{H}(\check{\mathbb{A}}^{+}) \bigoplus \mathbb{E}_{-}^{P}(\check{\mathbb{A}}^{+}),$$

like before, we call an eigenvalue of $\check{\mathbb{A}}^{+}$ hyperbolic if it vanishes for $\check{\zeta} = 0$ an parabolic otherwise. Remark well that, in this case, these denominations are sort of artificial since, by definition, $|\check{\zeta}| = 1$. $\mathbb{E}_{-}^{H}(\check{\mathbb{A}}^{+})$ and $\mathbb{E}_{-}^{P}(\check{\mathbb{A}}^{+})$ are then defined like before. The extended linear subspace $\mathbb{E}_{-}^{lim}(\mathbb{A}^{+})$ is then given by:

$$\mathbb{E}_-^{lim}(\mathbb{A}^+) = \mathbb{E}_-^H(\check{\mathbb{A}}^+)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0} \bigoplus \mathbb{E}_-^P(\mathbb{A}^+)|_{\zeta=0},$$

where $\mathbb{E}_-^H(\check{\mathbb{A}}^+)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0}$ stands for $\lim_{\check{\gamma}\to 0^+,\check{\tau}^2+\check{\gamma}^2=1} \lim_{\rho\to 0^+} \mathbb{E}_-^H(\check{\mathbb{A}}^+)(\check{\zeta},\rho)$. The same way, $\mathbb{E}_+(\mathbb{A}^-)$ extends continuously to $\mathbb{E}_+^{lim}(\mathbb{A}^-)$ as $\tilde{\zeta}$ goes to zero, with:

$$\mathbb{E}_+^{lim}(\mathbb{A}^-) = \mathbb{E}_+^H(\check{\mathbb{A}}^-)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0} \bigoplus \mathbb{E}_+^P(\mathbb{A}^-)|_{\zeta=0}.$$

The following Proposition shows the strong interest raised by the ability of computing explicitly $\mathbb{E}_-^{lim}(\mathbb{A}^+)$ and $\mathbb{E}_+^{lim}(\mathbb{A}^+)$.

Proposition 3.1. *Let us assume that the $(\tilde{\mathcal{H}}^\varepsilon, \tilde{\mathcal{M}})$ satisfies the Evans Condition which means that, for all $\zeta = (\tau,\gamma) \in \mathbb{R}\times\mathbb{R}^+ - \{0_{\mathbb{R}^2}\}$, there holds:*

$$\left|det\left(\tilde{\mathbb{E}}_-(\mathbb{A}^+(\zeta)), \tilde{\mathbb{E}}_+(\mathbb{A}^-(\zeta))\right)\right| > 0.$$

Then the four following properties are equivalent:

- *$(\tilde{\mathcal{H}}^\varepsilon, \tilde{\mathcal{M}})$ satisfies the **Uniform** Evans Condition.*
- *There is $\rho_0 > 0$ such that, for all $\zeta = (\tau,\gamma) \in \mathbb{R}\times\mathbb{R}^+ - \{0_{\mathbb{R}^2}\}$, with $|\zeta| < \rho_0$, there holds:*

$$\left|det\left(\mathbb{E}_-(\mathbb{A}^+(\zeta)), \mathbb{E}_+(\mathbb{A}^-(\zeta))\right)\right| \geq C > 0.$$

- $\left|det\left(\mathbb{E}_-^{lim}(\mathbb{A}^+), \mathbb{E}_+^{lim}(\mathbb{A}^-)\right)\right| > 0.$
- $\mathbb{E}_-^{lim}(\mathbb{A}^+) \bigcap \mathbb{E}_+^{lim}(\mathbb{A}^-) = \{0\}.$

Remark 3.2. *If we take $N = 1$ that is to say a scalar system, the uniform Evans condition is always satisfied. As a consequence, the uniform Evans condition also holds if A^+ and A^- are diagonalizable in the same basis.*

3.4 Analysis of the medium and high frequencies Evans function for 2×2 systems.

The bases in which A^+ and A^- are diagonal differ in general from each other. However, making the right change of basis, we can always assume that A^- is diagonal without loss of generality. Let us fix a positive real number K, for the Evans condition to hold, it is necessary that, for all $0 < |\zeta| < K$, the real and imaginary part of following determinant do not vanish simultaneously:

$$D(\zeta) = \begin{vmatrix} a & c & 1 & 0 \\ b & d & 0 & 1 \\ a\mu_1^+(\zeta) & c\mu_2^+(\zeta) & \mu_1^-(\zeta) & 0 \\ b\mu_1^+(\zeta) & d\mu_2^+(\zeta) & 0 & \mu_2^-(\zeta) \end{vmatrix}$$

where $\begin{pmatrix} a \\ b \end{pmatrix}$ is the normalized eigenvector associated to λ_1^+, which denotes the smallest eigenvalue of A^+ and $\begin{pmatrix} c \\ d \end{pmatrix}$ is the normalized eigenvector associated to λ_2^+, which is the greatest eigenvalue of A^+. We have thus $a^2+b^2=1$, $c^2+d^2=1$ and $ad-bc \neq 0$. Some computations show that:

$$D(\zeta) = (ad-bc)(\mu_1^+\mu_2^+ + \mu_1^-\mu_2^-) - ad(\mu_1^-\mu_2^+ + \mu_2^-\mu_1^+) + bc(\mu_2^-\mu_2^+ + \mu_1^-\mu_1^+)$$

Notice first that $Im(D(\zeta))$ does vanish for $\tau = 0$, thus a necessary condition in order for the Evans condition to hold is that $\Re e(D(0,\gamma))$ does not vanish for all γ positive. So, We will now study the sign of

$$\Re e D(\zeta) = D_1(\zeta) - D_2(\zeta)$$

where

$$\begin{gathered} D_1(\zeta) := ad(\Re e(\mu_1^+) - \Re e(\mu_1^-))(\Re e(\mu_2^+) - \Re e(\mu_2^-)) \\ +bc(\Re e(\mu_2^+) - \Re e(\mu_1^-)(\Re e(\mu_2^-) - \Re e(\mu_1^+)). \end{gathered}$$

and

$$\begin{gathered} D_2(\zeta) := ad(Im(\mu_1^+) - Im(\mu_1^-))(Im(\mu_2^+) - Im(\mu_2^-)) \\ +bc(Im(\mu_2^+) - Im(\mu_1^-)(Im(\mu_2^-) - Im(\mu_1^+)). \end{gathered}$$

Let us denote by $\lambda_1^+ < \lambda_2^+$ the two eigenvalues of A^+ and $\lambda_1^- < \lambda_2^-$ the two eigenvalues of A^-, we have then, for $i \in \{1;2\}$:

$$\mu_i^+ = \frac{1}{2}\lambda_i^+ - \frac{1}{4}((\lambda_i^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} \left(\left(1 + \frac{16\tau^2}{\left(\lambda_i^{+2}+4\gamma\right)^2} \right)^{-\frac{1}{2}} + 1 \right)$$

$$-i\ sign(\tau)\frac{1}{4}((\lambda_i^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} \left(1 - \left(1 + \frac{16\tau^2}{\left(\lambda_i^{+2}+4\gamma\right)^2} \right)^{-\frac{1}{2}} \right)$$

$$\mu_i^- = \frac{1}{2}\lambda_i^- + \frac{1}{4}((\lambda_i^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(\left(1+\frac{16\tau^2}{\left(\lambda_i^{-2}+4\gamma\right)^2}\right)^{-\frac{1}{2}}+1\right)$$

$$+i\ sign(\tau)\frac{1}{4}((\lambda_i^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tau^2}{\left(\lambda_i^{-2}+4\gamma\right)^2}\right)^{-\frac{1}{2}}\right)$$

As a consequence, restricting ourselves to $\tau = 0$ we have:

$$\mu_i^+|_{\tau=0} = \frac{1}{2}\left(\lambda_i^+ - ((\lambda_i^{+2}+4\gamma)^2)^{\frac{1}{4}}\right)$$

$$\mu_i^-|_{\tau=0} = \frac{1}{2}\left(\lambda_i^- + ((\lambda_i^{-2}+4\gamma)^2)^{\frac{1}{4}}\right).$$

Remark that, because A^+ and A^- are nonsingular, for all positive γ, there holds:

$$\mu_i^+|_{\tau=0} < 0,$$

$$\mu_i^-|_{\tau=0} > 0.$$

However, as γ vanishes, $\mu_i^+|_{\tau=0}$ or $\mu_i^-|_{\tau=0}$ may vanish too depending on the sign of λ_i^+ and λ_i^- .

3.5 Some sufficient assumptions for the Evans Condition to hold.

A necessary condition for the uniform Evans condition to hold is that, for all $\gamma > 0$, $|D(0,\gamma)| > 0$, which means that the sign of the following quantity remains strictly the same for all positive γ:

$$Q := ad(\mu_1^+|_{\tau=0} - \mu_1^-|_{\tau=0})(\mu_2^+|_{\tau=0} - \mu_2^-|_{\tau=0})$$

$$+bc(\mu_2^+|_{\tau=0} - \mu_1^-|_{\tau=0})(\mu_2^-|_{\tau=0} - \mu_1^+|_{\tau=0}) := Q_1 + Q_2.$$

For all $\gamma > 0$, we have thus

$$sign(Q_1) = sign(ad)$$

and

$$sign(Q_2) = -sign(bc).$$

Therefore, alternative sufficient conditions in order to obtain $|D(0,\gamma)| > 0, \quad \forall\gamma > 0$ are $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$. Indeed, as highlighted previously, for all nonzero ζ, $\mu_i^+|_{\tau=0} < 0$ and $\mu_i^-|_{\tau=0} > 0$. Our idea is, restricting ourselves to the cases where $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$, to search for sufficient conditions on the eigenvalues and eigenvectors of A^+ and A^- in order to ensure that $\Re e(D(\zeta))$ keeps the same sign as $D_1(\zeta)$ for all $\zeta \neq 0$. Take notice that, for all nonzero ζ, $D_1(\zeta)$ keeps strictly the same sign as $D_1|_{\tau=0}(\gamma)$, for $\gamma > 0$. Since $\Re e(D(\zeta)) = D_1(\zeta) - D_2(\zeta)$, if, for some ζ, $D_2(\zeta)$ is of opposite sign of $D_1(\zeta)$, we have to prove that $|D_2(\zeta)| < |D_1(\zeta)|$. The following lemma is useful in the study the sign of $\Re e D(\zeta)$:

Lemma 3.3. *Seeing μ^+ and μ^- as two functions of (ζ, λ), for all $\zeta \neq 0$, we have:*

$$Im(\mu^+(\zeta,\lambda)) = Im(\mu^+(\zeta,-\lambda)) = -Im(\mu^-(\zeta,\lambda)) = -Im(\mu^-(\zeta,-\lambda)).$$

Moreover

$$|Im(\mu^+(\zeta,\lambda))| < |\Re e(\mu^+(\zeta,\lambda))|,$$
$$|Im(\mu^-(\zeta,\lambda))| < |\Re e(\mu^-(\zeta,\lambda))|,$$

for all $\tau \neq 0$ and $\gamma \geq 0$.

Proof. The first part of this lemma is trivial, so let us prove the second part. For this purpose, let us fix $\gamma = \gamma_0$, we will then prove by an argument of comparative increasing speed in $|\tau|$ that for all $|\tau| > 0$, we have

$$|Im(\mu^\pm(\tau,\gamma_0,\lambda))| < |\Re e(\mu^\pm(\tau,\gamma_0,\lambda))|.$$

Let us begin by the study of μ^+. For all γ_0, there holds

$$|\Re e(\mu^+(0,\gamma_0,\lambda))| \geq |Im(\mu^+(0,\gamma_0,\lambda))| = 0,$$

and $|\Re e(\mu^+(\tau,\gamma_0,\lambda))|$, considered as a function of $|\tau|$, is increasing strictly quicker in $|\tau|$ than $|Im(\mu^+(\tau,\gamma_0,\lambda))|$, for all admissible value of (γ_0,λ), which proves the desired result. Indeed, we have:

$$|\Re e(\mu^+)| = -\frac{1}{2}\lambda^+ + \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} + \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{+2}+4\gamma)^2}\right)^{-\frac{1}{2}}$$

$$|Im(\mu^+)| = \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{+2}+4\gamma)^2}\right)^{-\frac{1}{2}}$$

If we fix the growth of $\frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}$ for increasing $|\tau|$ as a comparison state, the term $\frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{+2}+4\gamma)^2}\right)^{-\frac{1}{2}}$ is accelerating the growth of $|\Re e(\mu^+)|$ as $|\tau|$ gets bigger, but is delaying the growth of $|Im(\mu^+)|$. Noticing that:

$$|\Re e(\mu^-)| = \frac{1}{2}\lambda^- + \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} + \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{-2}+4\gamma)^2}\right)^{-\frac{1}{2}}$$

$$|Im(\mu^-)| = \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{-2}+4\gamma)^2}\right)^{-\frac{1}{2}}.$$

Reasoning the same way, we have thus proved that:

$$|Im(\mu^-(\zeta,\lambda))| < |\Re e(\mu^-(\zeta,\lambda))|.$$

$\square$

Theorem 3.4. *For $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$, the Evans condition always holds.*

Proof. We will begin by treating the case of medium frequencies. For $\tau = 0$, it has already been proven that the real part of the Evans function never vanishes and more precisely keeps the sign of ad or $-bc$ (take the non-null one by default). As a direct consequence of lemma 3.3, for all $\tau \neq 0$, there holds: $|\tau| > 0$ $\Re e(\mu_2^-) > |Im(\mu_2^-)| > 0$, $-\Re e(\mu_2^+) > |Im(\mu_2^+)| > 0$, $\Re e(\mu_1^-) > |Im(\mu_1^-)| > 0$, $-\Re e(\mu_1^+) > |Im(\mu_1^+)| > 0$. Thus, we have:

$$\Re e(\mu_1^-)\Re e(\mu_2^-) - Im(\mu_1^-)Im(\mu_2^-) \geq \Re e(\mu_1^-)\Re e(\mu_2^-) - |Im(\mu_1^-)||Im(\mu_2^-)| > 0,$$

$$\Re e(\mu_1^-)(-\Re e(\mu_2^+)) + Im(\mu_1^-)Im(\mu_2^+) \geq \Re e(\mu_1^-)(-\Re e(\mu_2^+)) - |Im(\mu_1^-)||Im(\mu_2^+)| > 0,$$
$$(-\Re e(\mu_1^+))\Re e(\mu_2^-) + Im(\mu_1^+)Im(\mu_2^-) \geq (-\Re e(\mu_1^+))\Re e(\mu_2^-) - |Im(\mu_1^+)||Im(\mu_2^-)| > 0,$$
$$(-\Re e(\mu_1^+))(-\Re e(\mu_2^+)) - Im(\mu_1^+)Im(\mu_2^+) \geq (-\Re e(\mu_1^+))(-\Re e(\mu_2^+)) - |Im(\mu_1^+)||Im(\mu_2^+)| > 0.$$

As a consequence, ad has the same sign as:

$$ad(\Re e(\mu_1^-) - \Re e(\mu_1^+))(\Re e(\mu_2^-) - \Re e(\mu_2^+)) - (Im(\mu_1^-) - Im(\mu_1^+))(Im(\mu_2^-) - Im(\mu_2^+)).$$

The same way, for all $\tau \neq 0$, $-bc$ has the same sign as:

$$bc(\Re e(\mu_1^-) - \Re e(\mu_2^+))(\Re e(\mu_1^+) - \Re e(\mu_2^-)) - bc(Im(\mu_1^-) - Im(\mu_2^+))(Im(\mu_1^+) - Im(\mu_2^-)).$$

Hence, assuming $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$, $\Re e D(\zeta)$ and thus $D(\zeta)$ does not vanish for all nonzero frequencies. The analysis performed here also works for high frequencies, where the eigenvalues $\mu^\pm$ of $\mathbb{A}^\pm$ have to be replaced by $\frac{\mu^\pm}{\Lambda}$, with $\Lambda > 0$, which ends our proof. $\square$

We have proved here Proposition 2.8 stated at the beginning of the paper. Remark that this Proposition states that the Evans Condition holds in some cases, without concern for the uniformity.

3.6 Some instances for which the uniform Evans condition does not hold.

This section is devoted to the proof of Proposition 2.9. We have shown during last section that the Evans condition always holds if $sign(ad) = -sign(bc)$. Consider (a,b,c,d) such that $ad - bc \neq 0$, $a^2 + b^2 = c^2 + d^2 = 1$, and $sign(ad) = sign(bc)$; $\lambda_1^- < \lambda_2^-$, $\lambda_1^+ < \lambda_2^+$. We shall search here for some $(a, b, c, d, \lambda_1^-, \lambda_1^+, \lambda_2^-, \lambda_2^+)$, inducing strong Evans-instabilities. More precisely, we will see that, upon correct choice of these parameters, $D|_{\tau=0}$ can vanish for some positive γ. To construct our example, we begin by making some sign assumptions on the eigenvalues corresponding to $q := \dim \Sigma = 0$:

$$\lambda_1^- < 0, \quad \lambda_2^- > 0, \quad \lambda_1^+ < 0, \quad \lambda_2^+ > 0.$$

For the sake of simplicity, we will assume that a, b, c, d are positive. Denoting by

$$D_a(\gamma) := ad(\Re e(\mu_1^+|_{\tau=0}) - \Re e(\mu_1^-|_{\tau=0}))(\Re e(\mu_2^+|_{\tau=0}) - \Re e(\mu_2^-|_{\tau=0})),$$

$$D_b(\gamma) := bc(\Re e(\mu_2^+|_{\tau=0}) - \Re e(\mu_1^-|_{\tau=0}))(\Re e(\mu_1^+|_{\tau=0}) - \Re e(\mu_2^-|_{\tau=0})),$$

we have $D|_{\tau=0} = D_a - D_b$. Note that $sign(D_a) = sign(D_b)$. Thus, $D|_{\tau=0}$ does not vanish for some $\gamma_0 > 0$ if and only if we have either $D_a > D_b$ for all positive γ, or $D_a < D_b$ for all positive γ. Observe that:

$$D_a(0) = ad(\lambda_1^+ - |\lambda_1^+| - \lambda_1^- - |\lambda_1^-|)(\lambda_2^+ - |\lambda_2^+| - \lambda_2^- - |\lambda_2^-|)$$

$$D_b(0) = bc(\lambda_2^+ - |\lambda_2^+| - \lambda_1^- - |\lambda_1^-|)(\lambda_1^+ - |\lambda_1^+| - \lambda_2^- - |\lambda_2^-|)$$

Due to the assumption we have made on the sign of the eigenvalues, we have:

$$D_a(0) = 4ad|\lambda_1^+||\lambda_2^-|,$$

$$D_b(0) = 0.$$

As a result, by continuity of D_a and D_b with respect to γ, we obtain that $D_a > D_b$ for γ in a positive neighborhood of zero. The interesting fact is that this inequality does not need any strong assumption to hold. Our goal will then be to prove that, for some $\gamma_0 > 0$, we have $D_a < D_b$, by continuity of D_a and D_b with respect to γ, this will prove the existence of a positive γ canceling the Evans function for $\tau = 0$. Remarking that D_a and D_b share some similarities in their constructions, we will take $\lambda_1^+ = -\lambda_2^+$ and $\lambda_1^- = -\lambda_2^-$ in order to build our example. By doing so, we have the simplified expressions of D_a and D_b:

$$D_a = ad\left(8\gamma + 2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} + 2\lambda_2^+\lambda_2^-\right)$$

$$D_b = bc\left(8\gamma + 2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} - 2\lambda_2^+\lambda_2^-\right).$$

Now take $bc = 2ad$, ($bc > ad$ would be sufficient to construct the example) denoting by $\gamma_0 := max\left(\frac{(\lambda_2^-)^2}{2}, \frac{(\lambda_2^+)^2}{2}\right)$, there holds $D_b(\gamma_0) > D_a(\gamma_0)$. Indeed,

$$D_b - D_a = bc\left(8\gamma + 2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} - 6\lambda_2^+\lambda_2^-\right),$$

and $2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} - 6\lambda_2^+\lambda_2^- \geq 0$ for all $\gamma \geq \gamma_0$. Thus, there is $0 < \gamma_1 < \gamma_0$ such that the Evans function vanishes for $\zeta = (0, \gamma_1)$.

3.7 Computation of the extension of the linear subspaces $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ and $\mathbb{E}_+^H(\check{\mathbb{A}}^-)$ in the case A^+ and A^- belongs to $\mathcal{M}_2(\mathbb{R})$.

Let us now inquire on a way to compute $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ and $\mathbb{E}_+^H(\check{\mathbb{A}}^-)$ for 2×2 systems. Due to the symmetry of the problem, we will only investigate the calculus of $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$. For small ρ, corresponding to $\check{\zeta}$ in a neighborhood ω of 0, let us look for an "hyperbolic" eigenvalue of $\check{\mathbb{A}}^+$ that we will note $\check{\lambda}^+(\check{\zeta}, \rho)$ in a generic manner, and compute its associated eigenvector:

$$\check{\mathbb{A}}^+(\check{\zeta}, \rho)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \check{\lambda}^+\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

Adopting the notation:

$$A^\pm := \begin{pmatrix} a_{11}^\pm & a_{12}^\pm \\ a_{21}^\pm & a_{22}^\pm \end{pmatrix}$$

we get, by multiplying some equations by $\rho > 0$ the following system:

$$\left\{\begin{array}{r} v_3 = \check{\lambda}^+ v_1, \\ v_4 = \check{\lambda}^+ v_2, \\ (i\check{\tau} + \check{\gamma})v_1 + a_{11}^+ v_3 + a_{12}^+ v_4 = \rho\check{\lambda}^+ v_3, \\ (i\check{\tau} + \check{\gamma})v_2 + a_{21}^+ v_3 + a_{22}^+ v_4 = \rho\check{\lambda}^+ v_4 \quad . \end{array}\right.$$

Making $\rho \to 0^+$ gives then, the following limiting system for low frequencies:

$$\left\{\begin{array}{r} v_3 = \check{\lambda}^+ v_1, \\ v_4 = \check{\lambda}^+ v_2, \\ (i\check{\tau} + \check{\gamma} + a_{11}^+ \check{\lambda}^+)v_1 + a_{12}^+ \check{\lambda}^+ v_2 = 0, \\ a_{21}^+ \check{\lambda}^+ v_1 + (i\check{\tau} + \check{\gamma} + a_{22}^+ \check{\lambda}^+)v_2 = 0 \quad . \end{array}\right.$$

Take notice that, in the above equation, $\check{\lambda}^+$ is also an unknown. In addition $\check{\lambda}^+ = 0$ is not an eigenvalue since it would imply that $v_1 = v_2 = v_3 = v_4 = 0$. To study the Asymptotic Evans function for low frequency in order to ensure that the Evans Condition holds uniformly, several cases would have to be treated. We will focus here, for some cases, on giving the way to compute the continuous extension of the subspaces to $\gamma = 0$, allowing then to check easily whether the uniform Evans Condition holds or not.
The dimension of the linear subspace $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is also p_+, the number of negative eigenvalues of A^+. We have then $\mathbb{E}_-^H(\check{\mathbb{A}}^+) = Span\left\{w_1^+, \dots, w_{p+}^+\right\}$.

The diagonal case where $a_{12}^+ = 0$ and $a_{21}^+ = 0$.

If $\lambda_j^+ = a_{jj}^+$ is a positive eigenvalue of A^+, then then one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} e_j \\ \check{\mu}_j^+ e_j \end{pmatrix}$, where e_j is the j^{th} vector of the canonical basis of $\mathbb{C}^2$ and $\check{\mu}_j^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_j^+}$.

The triangular case where $a_{12}^+ = 0$ and $a_{21}^+ \neq 0$.

$$\left\{\begin{array}{r} v_3 = \check{\lambda}^+ v_1, \\ v_4 = \check{\lambda}^+ v_2, \\ (i\check{\tau} + \check{\gamma} + a_{11}^+ \check{\lambda}^+)v_1 = 0, \\ a_{21}^+ \check{\lambda}^+ v_1 + (i\check{\tau} + \check{\gamma} + a_{22}^+ \check{\lambda}^+)v_2 = 0 \quad . \end{array}\right.$$

If $\lambda_2^+ = a_{22}^+$ is a positive eigenvalue of A^+, then one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} e_2 \\ \check{\mu}_2^+ e_2 \end{pmatrix}$, where e_2 is the second vector of the canonical basis of $\mathbb{C}^2$ and $\check{\mu}_2^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_2^+}$ is one of the eigenvalues with negative real part of $\check{\mathbb{A}}^+$. If $\lambda_1^+ = a_{11}^+$ is a positive eigenvalue of A^+, then $\check{\mu}_1^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_1^+}$ is one of the eigenvalues with negative real part of $\check{\mathbb{A}}^+$. The equation giving the associated eigenvectors is:

$$\begin{cases} v_3 = \check{\mu}_1^+ v_1, \\ v_4 = \check{\mu}_1^+ v_2, \\ v_1 \in \mathbb{C}, \\ v_2 = -\dfrac{a_{21}^+ \check{\mu}_1^+}{i\check{\tau} + \check{\gamma} + a_{22}^+ \check{\mu}_1^+} v_1 \end{cases} .$$

Hence one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} i\check{\tau} + \check{\gamma} + a_{22}^+ \check{\mu}_1^+ \\ -a_{21}^+ \check{\mu}_1^+ \\ \check{\mu}_1^+ \left(i\check{\tau} + \check{\gamma} + a_{22}^+ \check{\mu}_1^+ \right) \\ -a_{21}^+ \left(\check{\mu}_1^+ \right)^2 \end{pmatrix}$.

The triangular case where $a_{12}^+ \neq 0$ and $a_{21}^+ = 0$.

This case behaves similarly to the other triangular case just treated. If $\lambda_1^+ = a_{11}^+$ is a positive eigenvalue of A^+, then we can take

$$w_1^+ = \begin{pmatrix} e_1 \\ \check{\mu}_1^+ e_1 \end{pmatrix}$$

where e_1 is the first vector of the canonical basis of $\mathbb{C}^2$ and $\check{\mu}_1^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_1^+}$.
If $\lambda_2^+ = a_{22}^+$ is a positive eigenvalue of A^+, then one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} -a_{12}^+ \check{\mu}_2^+ \\ i\check{\tau} + \check{\gamma} + a_{11}^+ \check{\mu}_2^+ \\ -a_{12}^+ \left(\check{\mu}_2^+ \right)^2 \\ \check{\mu}_2^+ \left(i\check{\tau} + \check{\gamma} + a_{11}^+ \check{\mu}_2^+ \right) \end{pmatrix}$, where $\check{\mu}_2^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_2^+}$ is one of the eigenvalues with negative real part of $\check{\mathbb{A}}^+$.
These computations will allow us to conclude quickly the proof Proposition 2.10 done next section.

3.8 End of the proof of Proposition 2.10.

In view of the results proved until this section, we only lack the proof of the **uniform** nonvanishing of the Evans function as the frequencies come in a neighborhood of zero. For the examples given in Proposition 2.10, modulo a change of basis, we take:

$$A^- := \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix},$$

$$A^+ := \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix}$$

where d_1^-, d_2^-, d_1^+, d_2^+ and α are such that:
$\alpha \neq 0$ $d_1^- < 0$, $d_1^+ > 0$, $d_1^- \neq d_2^-$, and $d_1^+ \neq d_2^+$. Following Proposition 2.10 we will split our low frequency analysis of the Evans function into three parts depending on the signs of d_2^- and d_2^+.

The case $d_2^- < 0$ and $d_2^+ > 0$.

Note first that we are now considering a completely outgoing or expansive case, which implies that all the eigenvalues of $\mathbb{A}^+$ and $\mathbb{A}^-$ are hyperbolic. The computation of the asymptotic Evans function for low frequencies need the extension of the linear subspaces $\mathbb{E}_-(\mathbb{A}^+)$ and $\mathbb{E}_+(\mathbb{A}^-)$, which ceases to be well-defined as $|\zeta| \to 0$. Our problem satisfies our stability assumption (Uniform Evans Condition) iff the function D_{low} does not vanish for $\check{\gamma} = 0, \check{\tau} = 1$. D_{low} is defined as the modulus of the following determinant:

$$\begin{vmatrix} 1 & -\alpha\check{\mu}_2^+ & 1 & 0 \\ 0 & \nu_2^+ & 0 & 1 \\ \check{\mu}_1^+ & -\alpha(\check{\mu}_2^+)^2 & \check{\mu}_1^- & 0 \\ 0 & \check{\mu}_2^+\nu_2^+ & 0 & \check{\mu}_2^- \end{vmatrix}$$

We have thus:

$$D_{low} = |\nu_2^+||\check{\mu}_2^- - \check{\mu}_2^+||\check{\mu}_1^- - \check{\mu}_1^+|,$$

from which we get, since $|i\check{\tau} + \check{\gamma}| = 1$, that:

$$D_{low} = \left|1 - \frac{d_1^+}{d_2^+}\right| \left|-\frac{1}{d_2^-} + \frac{1}{d_2^+}\right| \left|-\frac{1}{d_1^-} + \frac{1}{d_1^+}\right| > 0.$$

Note well that, surprisingly D_{low} does not even depend of $\check{\zeta}$.

The case $d_2^- < 0$ and $d_2^+ < 0$.

We proceed like we have just done in the case where $d_2^- < 0$ and $d_2^+ > 0$. This time, thanks to the sign of d_2^+, $\mathbb{A}^+$ has one hyperbolic eigenvalue with negative real part that we will note $\check{\mu}_1^+$ and one parabolic eigenvalue with negative real part that we will note $\check{\mu}_2^+$. $\check{\mu}_1^+$ vanishes for $\tilde{\zeta} = 0$, whereas $\check{\mu}_2^+|_{\tilde{\zeta}=0} = d_2^+$. $\check{\mathbb{A}}^+$ has two eigenvalues with negative real parts:

$$\check{\mu}_1^+(\check{\zeta}) = -\frac{i\check{\tau} + \check{\gamma}}{d_1^+},$$

$$\check{\mu}_2^+(\check{\zeta}) = d_2^+.$$

As a consequence, we get that our problem satisfies our stability assumption (Uniform Evans Condition) iff the function D_{low} does not vanish for $\check{\gamma} = 0, \check{\tau} = 1$. D_{low} is defined as the modulus of the following determinant:

$$\begin{vmatrix} 1 & \alpha & 1 & 0 \\ 0 & d_2^+ - d_1^+ & 0 & 1 \\ \check{\mu}_1^+ & d_2^+\alpha & \check{\mu}_1^- & 0 \\ 0 & d_2^+(d_2^+ - d_1^+) & 0 & \check{\mu}_2^- \end{vmatrix}$$

We have thus:

$$D_{low} = |\check{\mu}_1^- - \check{\mu}_1^+||d_2^+ - d_1^+||\check{\mu}_2^- - d_2^+|,$$

from which we get, since $|i\check{\tau} + \check{\gamma}| = 1$, that:

$$D_{low} = \left|-\frac{1}{d_1^-} + \frac{1}{d_1^+}\right| |d_2^+ - d_1^+| \left(\left(\frac{\check{\tau}}{d_2^-}\right)^2 + \left(\check{\gamma} + d_2^+\right)^2\right).$$

Hence $D_{low}|_{\check{\tau}=1,\check{\gamma}=0} > 0$.

The case $d_2^- > 0$ and $d_2^+ > 0$.

This time $\check{\mathbb{A}}^+$ has two eigenvalues with negative real parts:

$$\check{\mu}_1^+(\check{\zeta}) = -\frac{i\check{\tau} + \check{\gamma}}{d_1^+},$$

$$\check{\mu}_2^+(\check{\zeta}) = -\frac{i\check{\tau} + \check{\gamma}}{d_2^+}.$$

As a consequence, we get that our problem satisfies our stability assumption (Uniform Evans Condition) iff the function D_{low} does not vanish for $\check{\gamma} = 0$ and $\check{\tau} = 1$. D_{low} is defined as the modulus of the following determinant:

$$\begin{vmatrix} 1 & -\alpha\check{\mu}_2^+ & 1 & 0 \\ 0 & \nu_2^+ & 0 & 1 \\ \check{\mu}_1^+ & -\alpha(\check{\mu}_2^+)^2 & \check{\mu}_1^- & 0 \\ 0 & \check{\mu}_2^+\nu_2^+ & 0 & \check{\mu}_2^- \end{vmatrix}$$

We have thus:

$$D_{low} = |\nu_2^+||\check{\mu}_1^- - \check{\mu}_1^+||\check{\mu}_2^- - \check{\mu}_2^+|;$$

hence, since $|i\check{\tau} + \check{\gamma}| = 1$, we obtain:

$$D_{low} = |i\check{\tau} + \check{\gamma} + d_1^+\check{\mu}_2^+||\check{\mu}_1^- - \check{\mu}_1^+||d_2^- - \check{\mu}_2^+|$$

and then

$$D_{low} = \left|1 - \frac{d_1^+}{d_2^+}\right|\left|-\frac{1}{d_1^-} + \frac{1}{d_1^+}\right|\left(\left(d_2^- + \frac{\check{\gamma}}{d_2^+}\right)^2 + \left(\frac{\check{\tau}}{d_2^+}\right)^2\right) > 0.$$

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